Khalifa University

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Wave Modulation in Nonlinear Lattices: Nonlinear Schrödinger Formalism and Application in Dusty Plasma Crystals

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1. Introduction & Modeling

We have undertaken a study of the modulational dynamics of wave packets propagating in a one-dimensional (1D) hybrid Fermi-Pasta-Ulam-Tsingou / Klein-Gordon type lattice chain, incorporating an arbitrary polynomial (i.e. quadratic and/or cubic) coupling anharmonicity, in the presence of a nonlinear on-site (substrate) potential.

Consider a nonlinear lattice (chain) whose n^{th} site has the equation of motion:

$$\begin{aligned} m\,\ddot{\psi}_n =& \mathcal{K}(\psi_{n+1}+\psi_{n-1}-2\psi_n) \\ &+ \mathcal{K}\,\beta\left[(\psi_{n+1}-\psi_n)^3-(\psi_n-\psi_{n-1})^3\right] \\ &- m\left(\omega_0^2\psi_n + a\psi_n^2 + b\psi_n^3\right)\ , \end{aligned} \qquad (1)$$

where, $\mathcal{K}$ is the spring constant, m is the mass and ψ_n is the displacement of mass at site n.

The quasi-continuum approximation, viz. setting $\psi_n(t) \simeq \psi(x,t)$, and Taylor expanding, $\psi(n\pm D, t) \approx \psi \pm D\psi_x + \frac{1}{2}D^2\psi_{xx}\cdots$

$$\psi_{tt} - c_0^2\psi_{xx} + \omega_0^2\psi = 3\beta c_0^2 D^2(\psi_x)^2\psi_{xx} - a\psi^2 - b\psi^3\,. \qquad (2)$$

Here, D is the lattice constant, and we have defined the speed $c_0^2 = \mathcal{K}(D^2/m)$.

2. Application to dusty plasma crystals

This research applies directly to the modeling of transverse dust-lattice waves in dusty plasma crystals [1, 2].

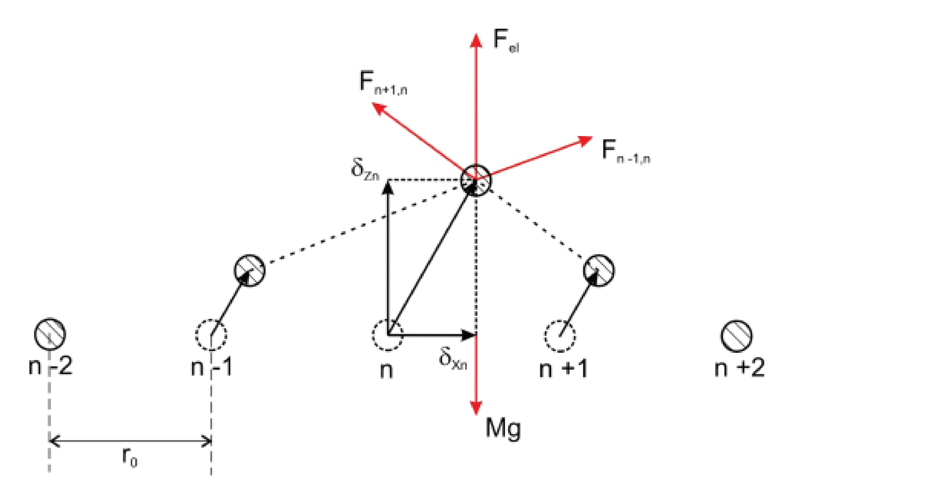

Figure 1: Dust grain vibrations in the longitudinal ($\sim \hat{x}$) and transverse ($\sim \hat{z}$) directions, in a ($1d$) dust lattice. Note that the longitudinal degree of freedom is assumed to be "frozen" in this work. Figure adapted from [1].

It has been shown in earlier studies [1] that the equation of motion for nonlinear transverse dust lattice waves (TDLW) reads:

$$\begin{aligned} \frac{d^2 z_n}{dt^2} =& \omega_{0,T}^2(2z_n - z_{n+1} - z_{n-1}) \\ &+ \frac{a_{02}}{r_0}\left[(z_{n+1}-z_n)^3 - (z_n - z_{n-1})^3\right] \\ &- \omega_g^2 z_n - a z_n^2 - b z_n^3\,. \end{aligned} \qquad (3)$$

Here, the coupling coefficients $\omega_{0,T}$ and a_{02} depend on the plasma Debye length λ_D as,

$$\omega_{0,T}^2 = \frac{q^2}{M\lambda_D^3}\,e^{-\kappa}\frac{1+\kappa}{\kappa^3}\,, \qquad \frac{a_{02}}{r_0} = \frac{q^2}{2M\lambda_D^5}\,e^{-\kappa}\frac{\kappa^2+3\kappa+3}{\kappa^5}\,,$$

$$\kappa = \frac{r_0}{\lambda_D}\,, \qquad (4)$$

here, q is the charge of the dust particles (assumed to be constant), r_0 is the distance between dust grains, M is the mass of dust particles and z_n is the vertical displacement of the n^{th} dust grain. The parameters ω_g, a and b are found experimentally [1, 3].

In the continuum approximation, (3) will becomes:

$$z_{tt} + c_T^2 z_{xx} + \omega_g^2 z = 3a_{02}\,r_0^3(z_x)^2 z_{xx} - a z^2 - b z^3\,, \qquad (5)$$

where we used $\omega_{0,T}^2\, r_0^2 = c_T^2$. Upon a simple comparison, we note that Eq. (5) is formally identical to Eq. (2), upon a straightforward change in notation.

3. NLSE Framework for Dust Lattices

Applying Newell's multiple scales technique in its quasi-discrete version (see details in [4]), we find a solution representing a modulated wavepacket:

$$\begin{aligned} z_n(t) \approx& \epsilon\left[\hat{\psi}\,e^{i(knr_0-\omega t)} + c.c\right] \\ &+ \varepsilon^2 a\left[-\frac{2|\hat{\psi}|^2}{\omega_g^2} + \frac{\psi^2}{3\omega_g^2}e^{2i(knr_0-\omega t)} + c.c\right] + \mathcal{O}(\varepsilon^3) \end{aligned} \qquad (6)$$

where c.c. denotes the complex conjugate.

The first-harmonic amplitude ψ is given by the NLS equation:

$$i\frac{\partial\psi}{\partial T} + P\frac{\partial^2\psi}{\partial X^2} + Q|\psi|^2\psi = 0\,, \qquad (7)$$

where $X = \epsilon(x - v_g t)$, $T = \epsilon^2 t$ (assuming $\epsilon \ll 1$) and $v_g = \omega'(k)$ denotes the (negative, in this case) group velocity.

The rescaled (dimensionless) coefficients are given by,

$$P = -e^{-\kappa}\frac{1+\kappa}{\kappa}\frac{1}{2\omega^3} \;<\; 0\,\forall\,k\,, \qquad (8)$$

$$Q = \frac{1}{2\omega}\left(\frac{10a^2}{3} - 3b - e^{-\kappa}\frac{\kappa^2+3\kappa+3}{\kappa}k^4\right)\,. \qquad (9)$$

In deriving the above, we have rescaled the algebraic expressions above by normalizing time by ω_g^{-1} and length by c_T/ω_g.

4. Focusing ($PQ > 0$) vs defocusing ($PQ < 0$) regime

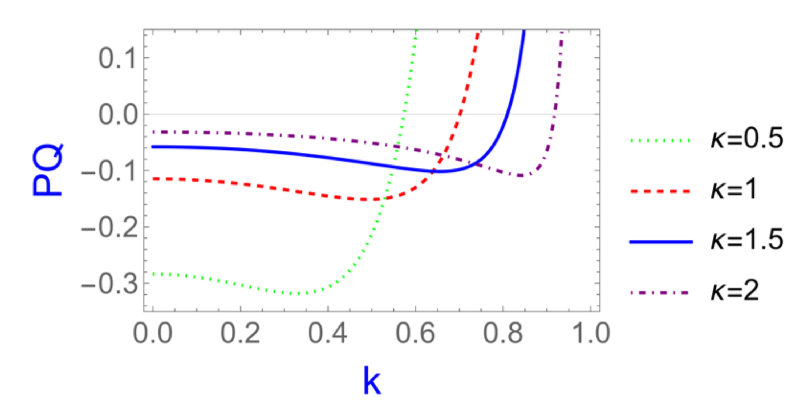

Figure 2: The product PQ is depicted versus the carrier wavenumber k , for various κ values. Typical values adapted from [3] are: $a \approx -0.5$ and $b \approx 0.07$.

5. Modulational Instability

When $PQ > 0$, Benjamin–Feir modulational instability occurs, with growth rate given by :

$$\Gamma = |P\,\nu|\sqrt{\nu_{crit}^2 - \nu^2}\,. \qquad (10)$$

Here, the critical wavenumber (ν_{crit}) is defined as:

$$\nu_{crit} = A_0\sqrt{\frac{2Q}{P}}\,,$$

where, A_0 is the wavepacket amplitute. The maximum growth rate of Γ occurs at : $\nu_{max} = \nu_{crit}/\sqrt{2}$.

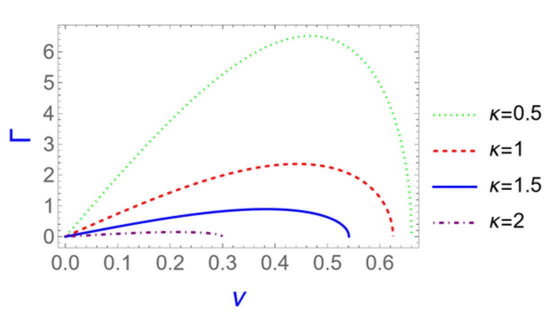

Figure 3: Modulational instability growth for varying κ values for arbitrary values of $k = 0.95$ and $A_0 = 1$.

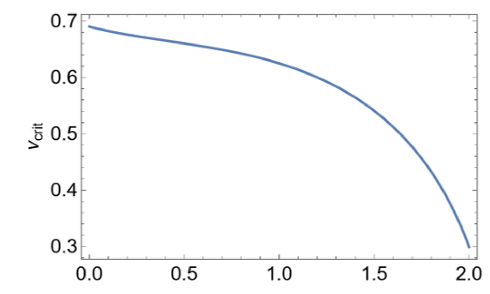

Figure 4: The line represents ν_{crit} vs κ for arbitrary values of $k = 0.95$ and $A_0 = 1$.

6. Envelope soliton solutions of the NLS Equation

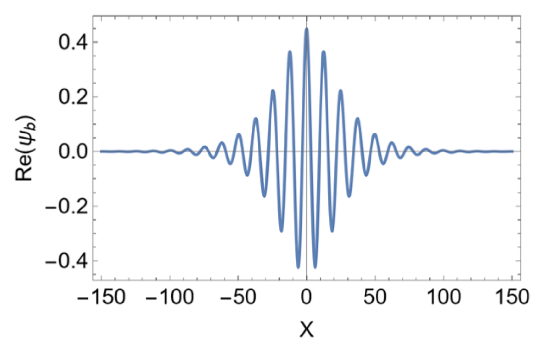

Figure 5: Bright envelope soliton for $\kappa = 1.5$, $k = 0.82$, $u_e = 1$, $\rho_0 = 0.2$, $\Omega = 1$, $T = 0$.

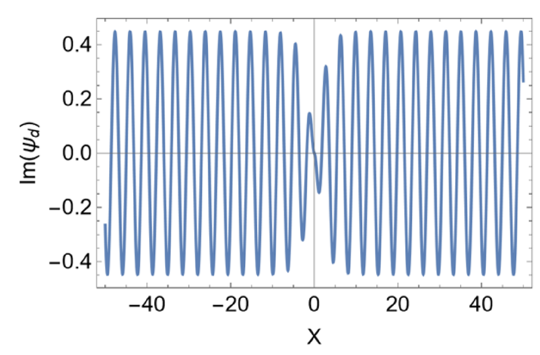

Figure 6: Dark envelope soliton for $\kappa = 1.5$, $k = 0.5$, $u_e = 1$, $\rho_0 = 0.2$, $T = 0$.

The Peregrine solution is given by (expression adapted from [5]) :

$$\hat{\psi} = \frac{\sqrt{p_0}}{Q}\left[1 - \frac{4\left(1+2i\frac{p_0}{Q}T\right)}{1+4\frac{p_0}{Q}T^2 + \frac{p_0}{PQ}X^2}\right]e^{i\frac{p_0}{Q}T}\,, \qquad (11)$$

where $p_0 \in \mathbb{R}_+^*$.

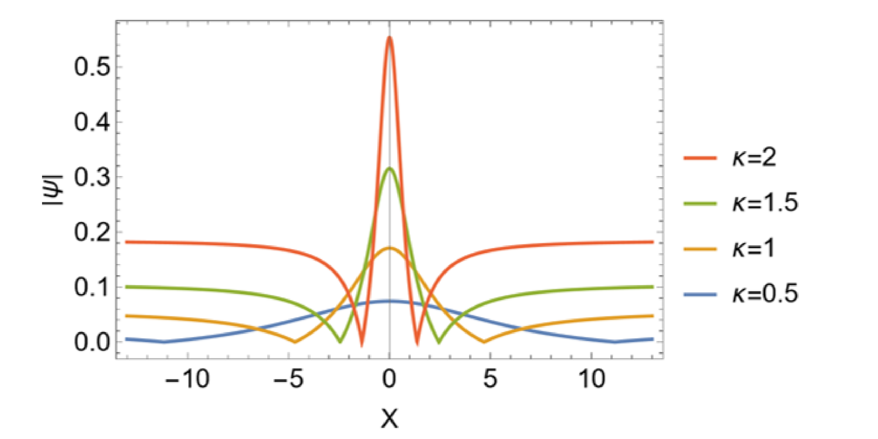

Figure 7: Peregrine soliton for $k = 1.5$, $p_0 = 0.01$ and $T = 0$.

7. Conclusions

- Study: Preliminary investigation of energy localization in TDLW in dusty plasma crystals.
- Analyzed how focusing vs. defocusing regimes shift with change in κ.
- Predictions include: envelope modes, MI, and rogue wave solutions.

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