CHAOS SYMBOLIC DYNAMICS AND MELNIKOV THEORY

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$$f^k(x) = f(f(f(\ldots(\mathrm{f}(\mathrm{x}))..)$$

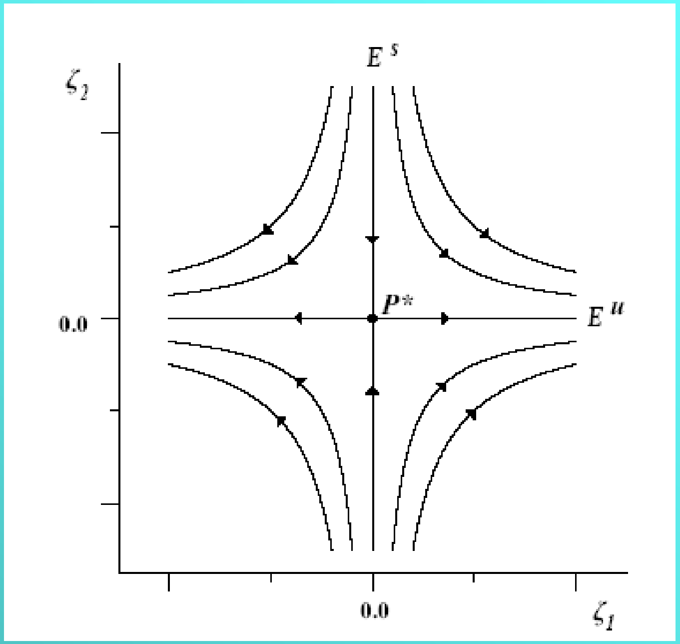
Maps in $\mathbb{R}^2$

$$f:\mathbb{R}^2 \to \mathbb{R}^2 \qquad x_{n+1} = f(x_n)$$

Fixed point: $$x_{n+1} = f(x_n) = x_n$$

Periodic point: $$x_{n+k} = f^k(x_n) = x_n$$

Saddle point



Stable manifold $W^s = \{p \in \mathbb{R}^2 : f^k(p) \to P* \text{ as } k \to +\infty\}$

Unstable manifold $W^u = \{p \in \mathbb{R}^2 : f^k(p) \to P* \text{ as } k \to -\infty\}$

Maps in $\mathbb{R}^2$

$$f : \mathbb{R}^2 \to \mathbb{R}^2 \qquad x_{n+1} = f(x_n)$$

Fixed point: $x_{n+1} = f(x_n) = x_n$

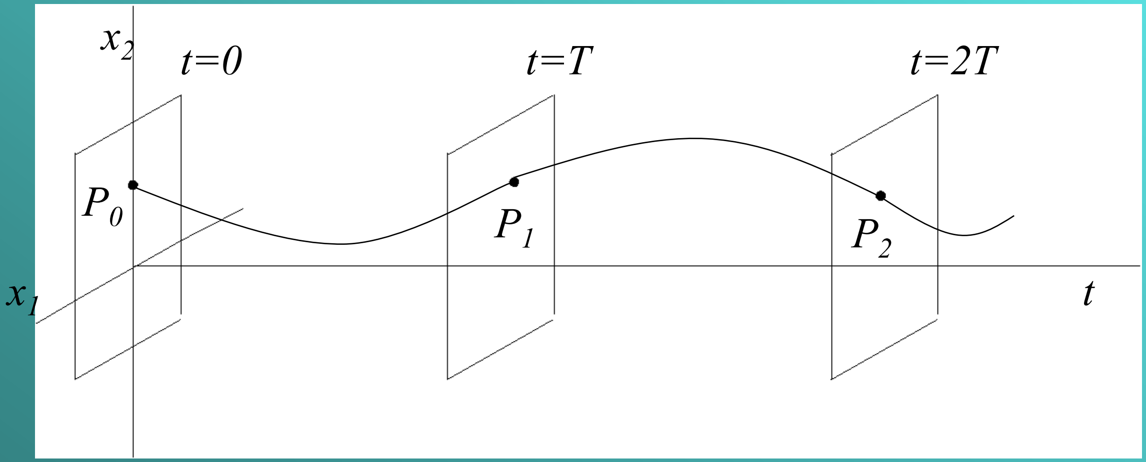
Periodic point: $x_{n+k} = f^k(x_n) = x_n$

Non-autonomous systems in $\mathbb{R}^2$ with periodic time-dependence

$$\dot{x}_i = g_i(x_i, t) \quad i = 1, 2$$

$$g_i(x_i, t+T) = g_i(x_i, t)$$

The Poincaré map



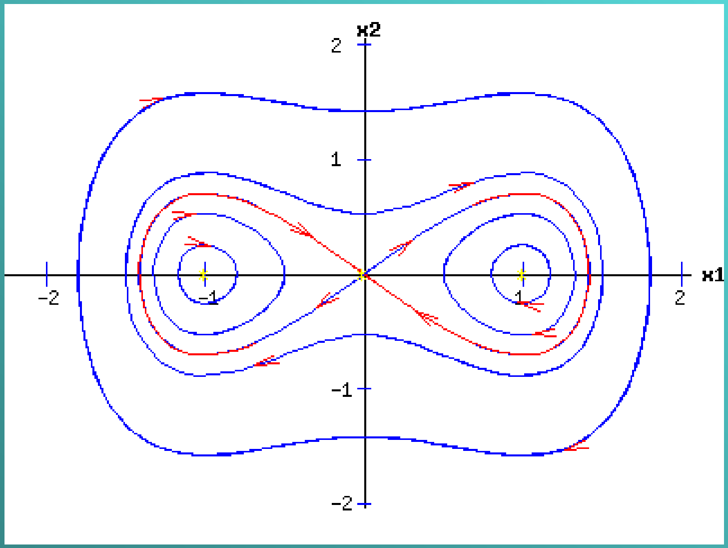
Perturbed system

$$\dot{x}_1 = \frac{\partial H(x_i)}{\partial x_2} + \varepsilon f_1(x_i, t)$$

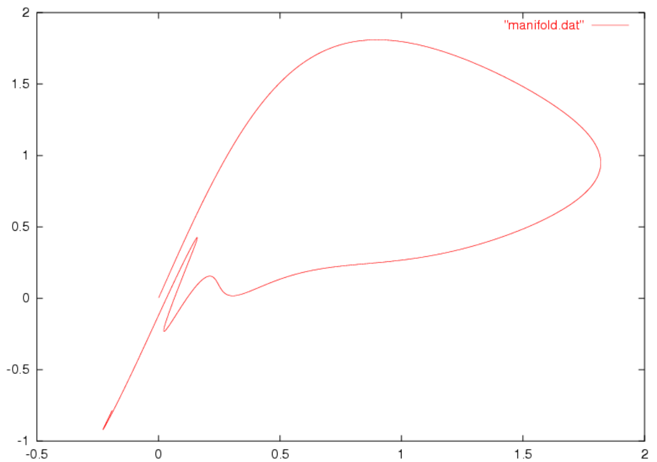
$$\dot{x}_2 = -\frac{\partial H(x_i)}{\partial x_1} + \varepsilon f_2(x_i, t)$$

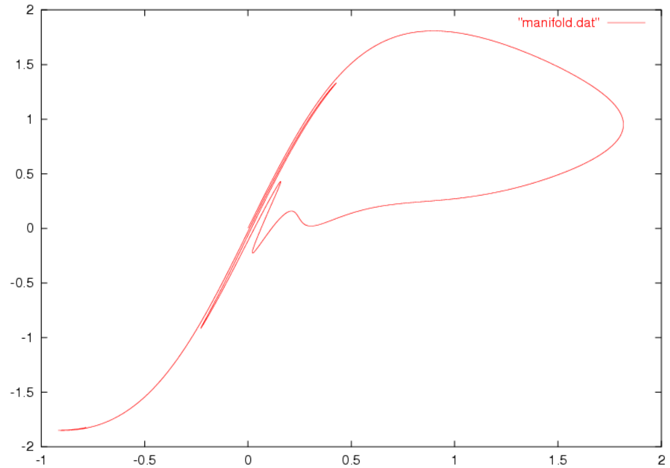
$$f_i(x_i, t+T) = f_i(x_i, t)$$

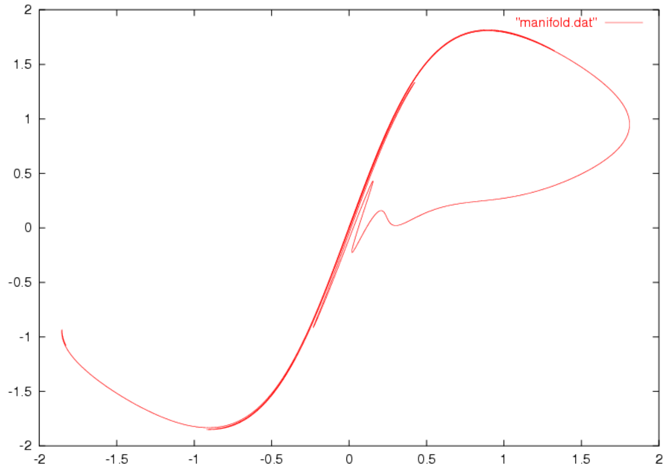
Phase space and Poincaré map of the unperturbed system (ε=0)

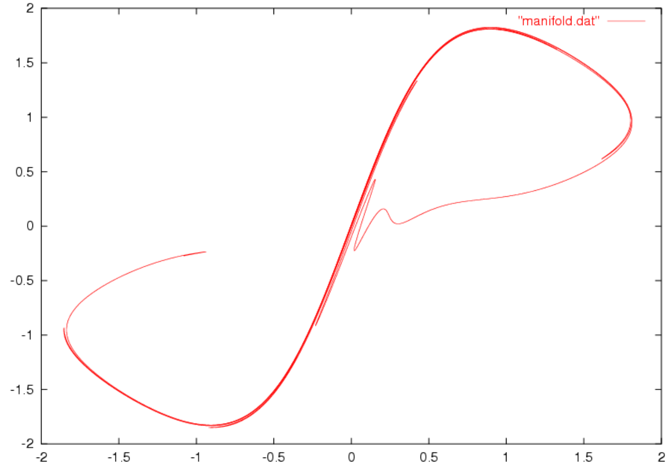


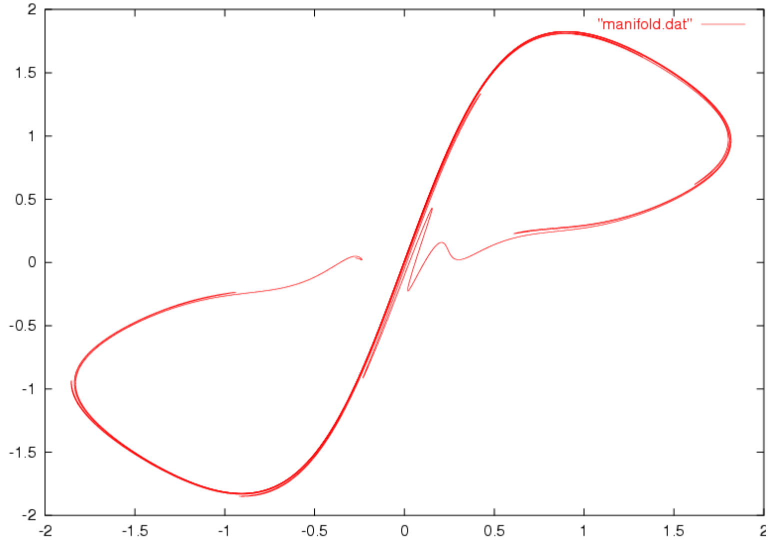
The Poincaré map of the perturbed system







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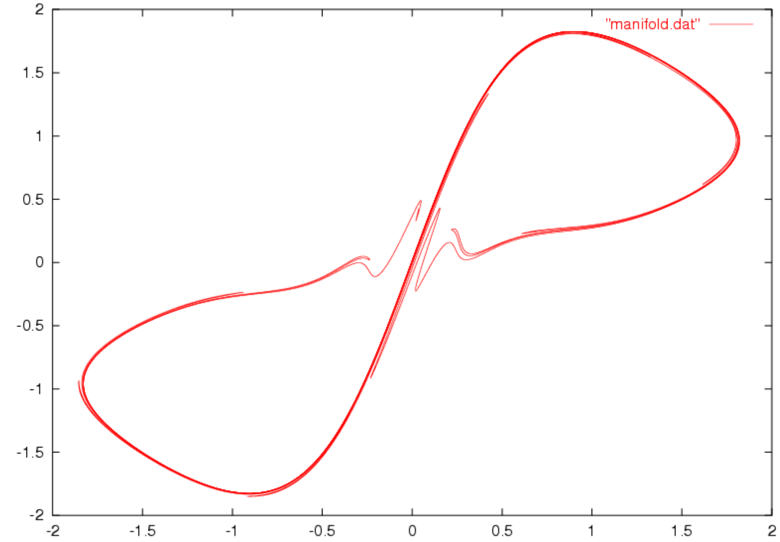
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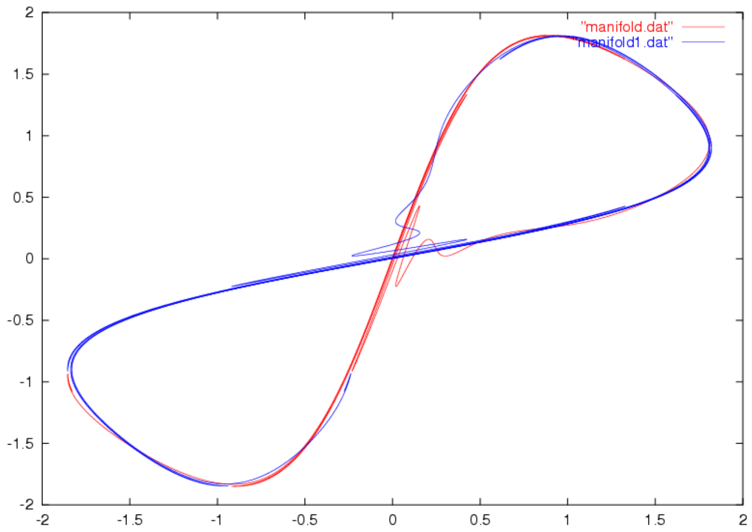
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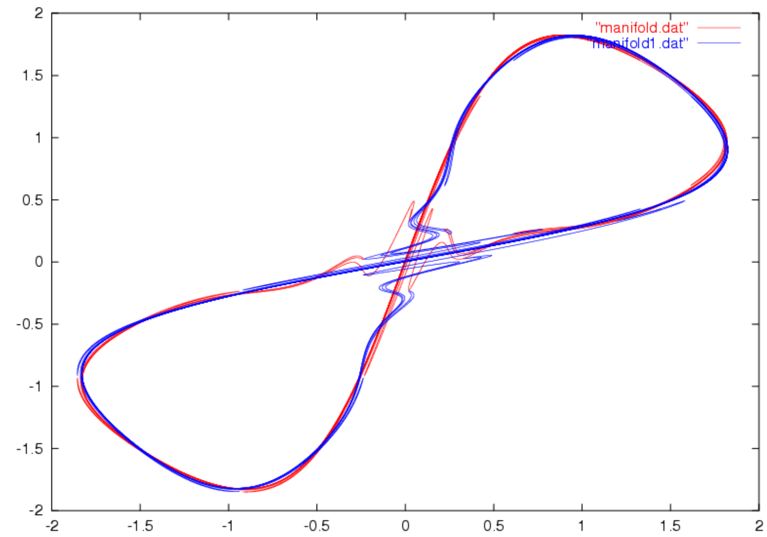
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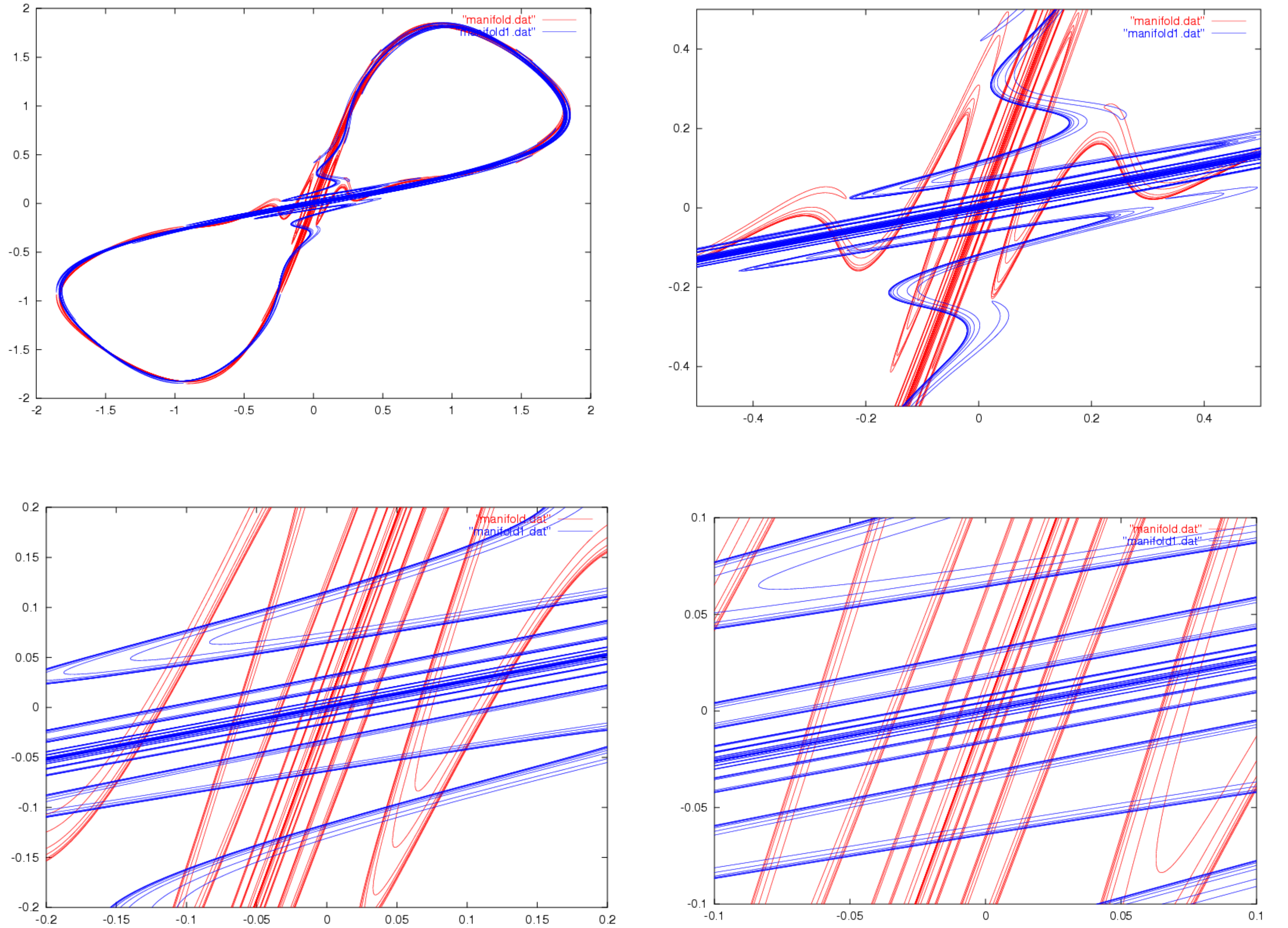
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"manifold.dat"
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-0.05
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Chaotic invariant sets

Invariant set $S = f(S)$

The **compact** invariant set S is **chaotic** if:

- It has a **dense set** of periodic points
- It is **topologically transitive**

Then it can be shown that:

- It has **sensitive dependence on initial conditions**

Compact set: For subsets of $\mathbb{R}^2$, closed and connected

Dense set: *A* is dense in *B*, if for every point $x \in B$

there is a sequence of points of *A* that tends to *x*

Topological transitive mapping in the in invariant set *S*:

$$\forall U, V \in S \ \exists \ k \in \mathbb{N}: \quad f^k(U) \cap V \neq \emptyset$$

Sensitive dependence on initial conditions:

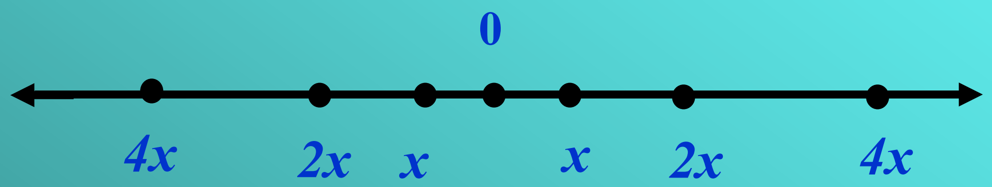
$\exists \ r > 0 : \forall x \in S$ and $\forall \varepsilon > 0 \ \exists \ y \in S: d(x, y) < \varepsilon$ and $k \in \mathbb{N}$:
$d(f^k(x), f^k(y)) > r$

Counterexamples

1. The map

$$x \to 2x \qquad x \in \mathbb{R}$$

is **regular**. Fixed point: $x = 0$



All initial conditions tend to infinity. An initial uncertainty Δx_0 increases exponentially

$$\Delta x_k = 2^k \Delta x_0$$

but there is no **mixing** of states.

2. All points in the map

$$\phi \to \phi + a \quad \mathrm{mod}\,1 \quad \phi \in S^1,\ a \in \mathbb{Q}$$

Are **periodic**.

If $\alpha = p/q$ then

$$f^q(\varphi) = \phi + qa = \phi + p = \varphi \quad \mathrm{mod}\,1$$

There is no topological transitivity or sensitive dependence on initial conditions

$$f(\varphi) - f(\varphi') = \varphi - \varphi'$$

3. In the map

$$\phi \to \phi + a \quad \mathrm{mod}\,1 \quad \phi \in S^1,\ a \in \mathbb{R}\backslash\mathbb{Q}$$

every orbit is **dense** in S^1.

$$\forall \varepsilon > 0\ \exists k \in \mathbb{Z}^+: \quad |\phi - f^k(\phi)| < \varepsilon \quad \mathrm{mod}\,1$$

and the sequence of points

$$f^k(\varphi), f^{2k}(\varphi), \ldots, f^{mk}(\varphi), \ldots$$

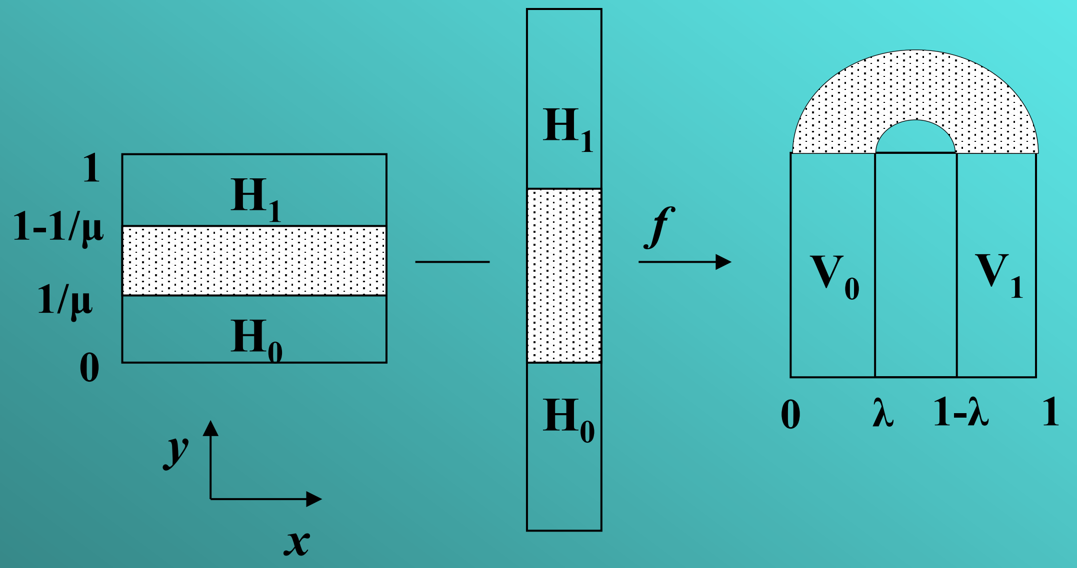
divides the circle in arcs of length less than ε

The map is **topologically transitive** but has no periodic points nor sensitive dependence on initial conditions

The Smale horseshoe

We consider the map $f: D \to \mathbb{R}^2$

where D is the unit square:

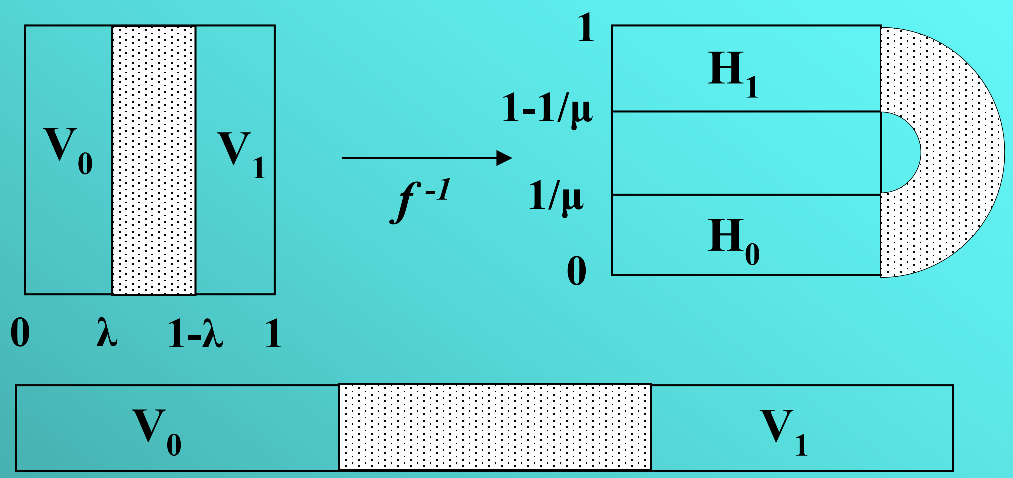


The map acts on H_0 as follows:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

and on H_1 as:

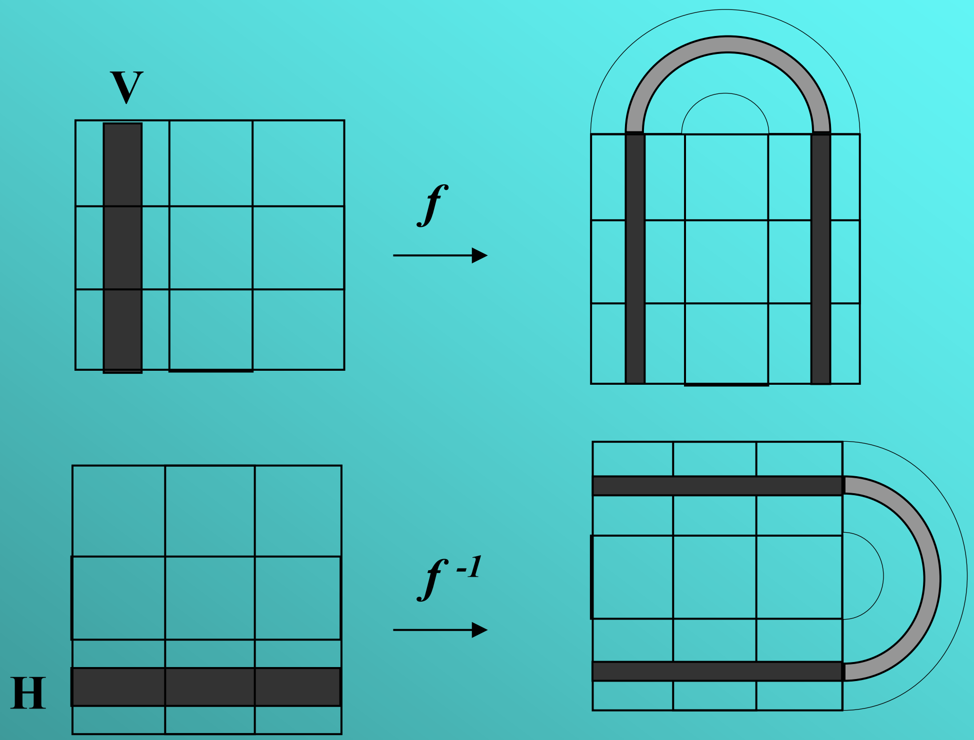
$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} -\lambda & 0 \\ 0 & -\mu \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} 1 \\ \mu \end{pmatrix}$$

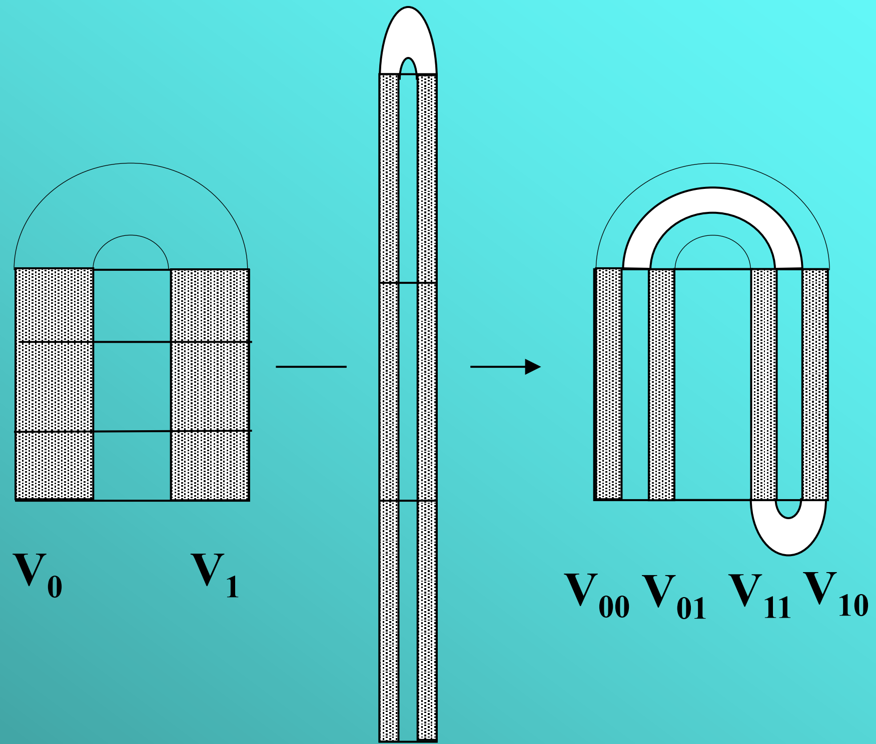


$$f(H_0) = V_0, \quad f(H_1) = V_1$$

$$f^{-1}(V_0) = H_0, \quad f^{-1}(V_1) = H_1$$

Lemma



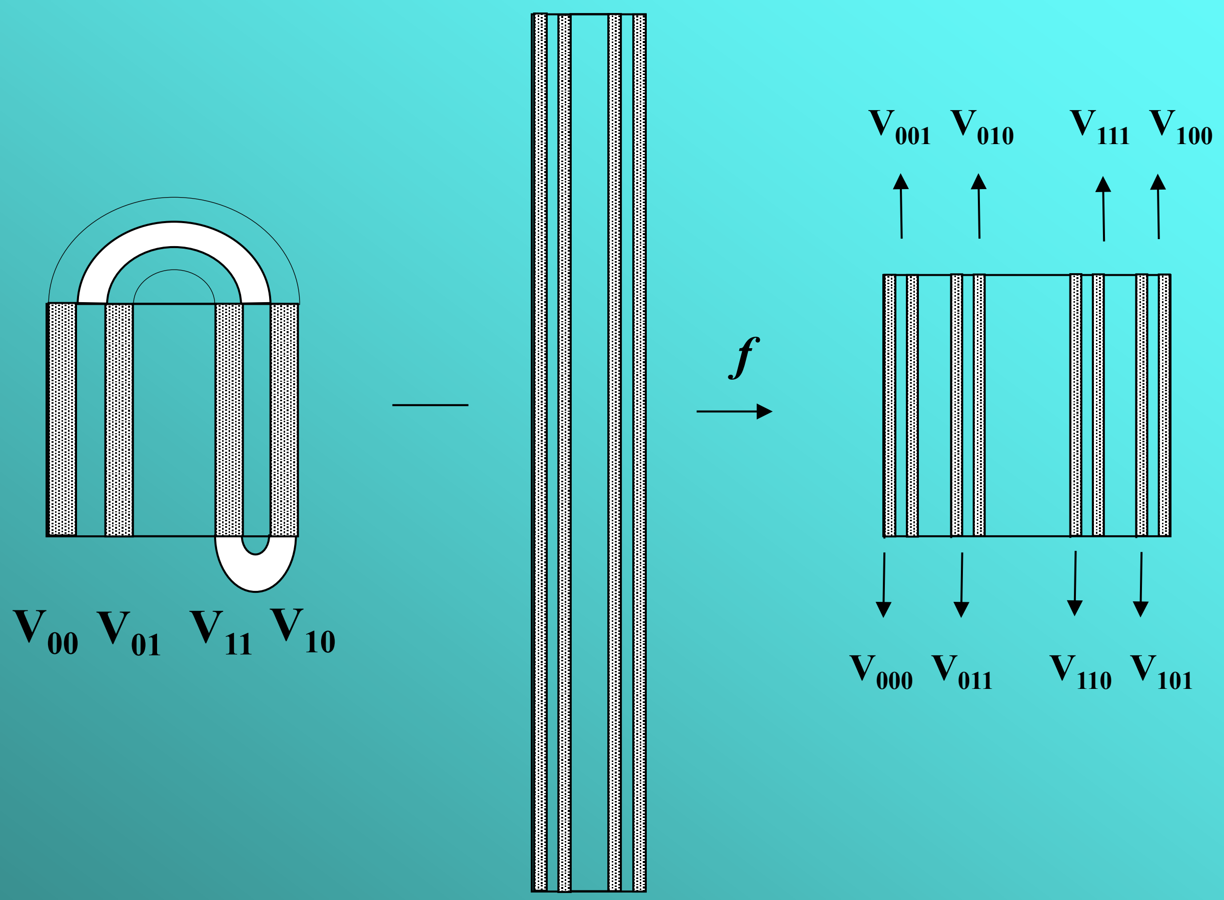


1. $D \cap f(D) = V_0 \cup V_1$

2. $D \cap f(D) \cap f^2(D) = \bigcup_{s_{-1}s_{-2}} V_{s_{-1}s_{-2}}$

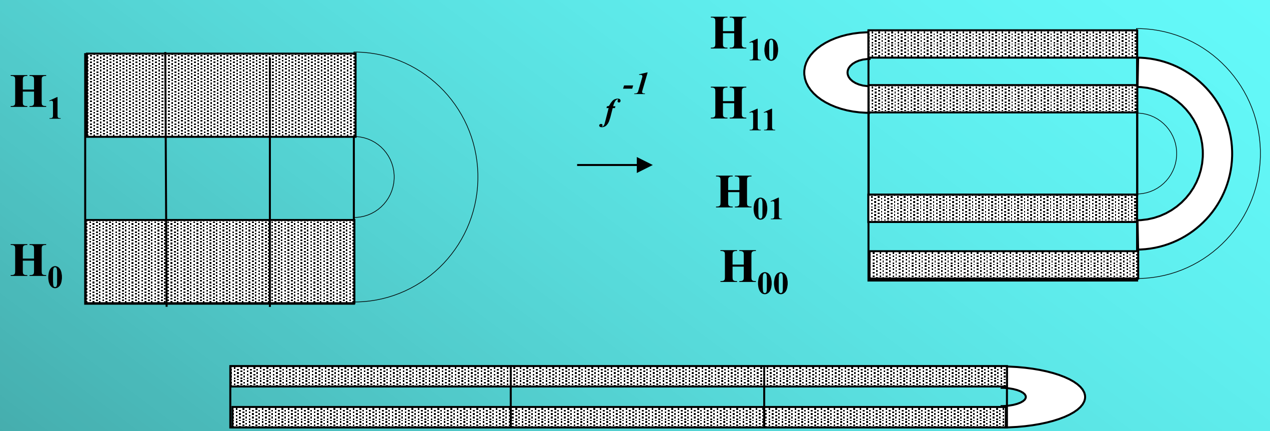
....

k. $D \cap f(D) \cap \ldots \cap f^k(D) = \bigcup_{s_{-1}s_{-2}\ldots s_{-k}} V_{s_{-1}s_{-2}\ldots s_{-k}}$



1. $V_{s_{-1}s_{-2}\ldots s_{-k}} \subset V_{s_{-2}\ldots s_{-k}}$

2. $V_{s_{-1}s_{-2}\ldots s_{-k}} = \left\{ p \in D : f^{-i+1}(p) \in V_{s_{-i}},\, i = 1, \ldots, k \right\}$

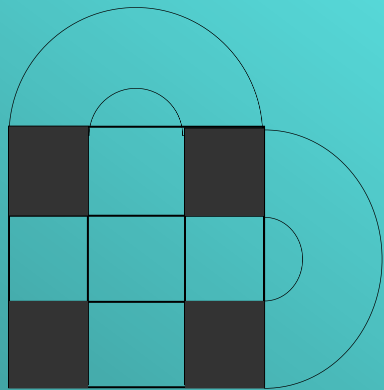


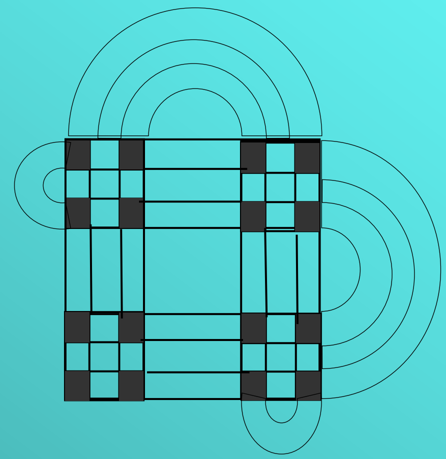
1. $D \cap f^{-1}(D) = H_0 \cup H_1$
2. $D \cap f^{-1}(D) \cap f^{-2}(D) = \bigcup_{s_0 s_1} H_{s_0 s_1}$

....

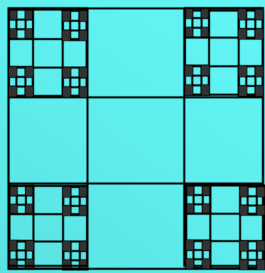
k. $D \cap f^{-1}(D) \cap \ldots \cap f^{-k}(D) = \bigcup_{s_0 s_1 \ldots s_k} H_{s_0 s_1 \ldots s_k}$

1. $H_{s_0 s_1 \ldots s_k} \subset H_{s_1 \ldots s_k}$
2. $H_{s_0 s_1 \ldots s_k} = \left\{ p \in D : f^i(p) \in H_{s_i},\, i = 0, \ldots, k \right\}$





$\Lambda_2 \cap \Lambda_1 \cap \Lambda_{-1} \cap \Lambda_{-2}$



$\Lambda_3 \cap \Lambda_2 \cap \Lambda_1 \cap \Lambda_{-1} \cap \Lambda_{-2} \cap \Lambda_{-3}$

Symbolic Dynamics

$$\Lambda = \bigcap_{k=-\infty}^{\infty} f^k(D)$$

This is an invariant set of the map which is a **Cantor set** and **chaotic**.

There is an 1-1 correspondence between $p \in \Lambda$ and doubly infinite series of two symbols

$$s = \ldots s_{-k} \ldots s_{-1} . s_0 s_1 \ldots s_k \ldots$$

We define in **Σ** the metric

$$d(s, s') = \sum_{k=-\infty}^{\infty} \frac{| s_k - s'_k |}{2^k}$$

The map $\varphi : \Lambda \rightarrow \Sigma$ is a **homeomorphism** i.e. it is

1. One to one
2. Continuous
3. The inverse is also continuous

Topological conjugacy

$$\begin{array}{ccc} \Lambda & \xrightarrow{f} & \Lambda \\ \varphi \downarrow & & \downarrow \varphi \\ \Sigma & \underset{\sigma}{\longrightarrow} & \Sigma \end{array}$$

The maps f and σ are topologically conjugate through φ i.e.

$$f \circ \varphi = \varphi \circ \sigma$$

and therefore if $\boldsymbol{\sigma}$ is chaotic in $\boldsymbol{\Sigma}$, then $\boldsymbol{f}$ is chaotic in the invariant set $\boldsymbol{\Lambda}$.

$\boldsymbol{\sigma}$ is chaotic in $\boldsymbol{\Sigma}$

1. It has a dense set of periodic points

$$\sigma^{2k+1}(\overline{s_{-k}\dots s_{-1}.s_0 s_1\dots s_k}) = \overline{s_{-k}\dots s_{-1}.s_0 s_1\dots s_k}$$

2. It is topologically transitive

$$s = \dots s_{-k}\dots s_{-1}.s_0 s_1\dots s_k\dots \in U$$

$$s' = \dots s'_{-m}\dots s'_{-1}.s'_0 s'_1\dots s'_m\dots \in V$$

Then for the point

$$s'' = \dots s_{-k}\dots s_{-1}.s_0 s_1\dots s_k s'_{-m}\dots s'_{-1}s'_0 s'_1\dots s'_m\dots \in U$$

$$\sigma^{k+m+1}(s'') \in V$$

3. It has sensitive dependence on initial conditions

The points

$$s = \ldots s_{-k-1} s_{-k} \ldots s_{-1} . s_0 s_1 \ldots s_k s_{k+1} \ldots,$$

$$s' = \ldots s_{-k} \ldots s_{-1} . s_0 s_1 \ldots s_k s'_{k+1} \ldots$$

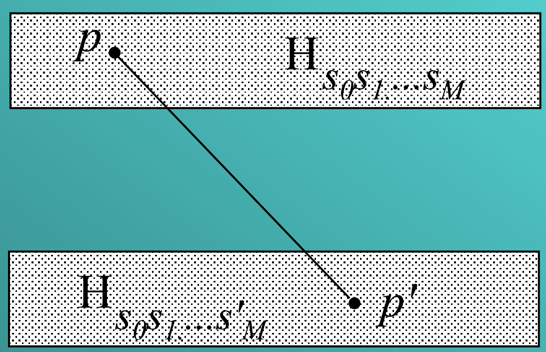
are initially ε-close while

$$d(\sigma^{k+1}(s), \sigma^{k+1}(s')) \geq 1$$

Λ is a Cantor set

A set $\Lambda \subset \mathbb{R}^2$ is a **Cantor set** if it is

1. **Closed**: It contains all its accumulation points
2. **Perfect**: All points of Λ are accumulation points
3. **Totally disconnected**: It does not contain any open interval



The homoclinic Mel'nikov theory

We consider again the system

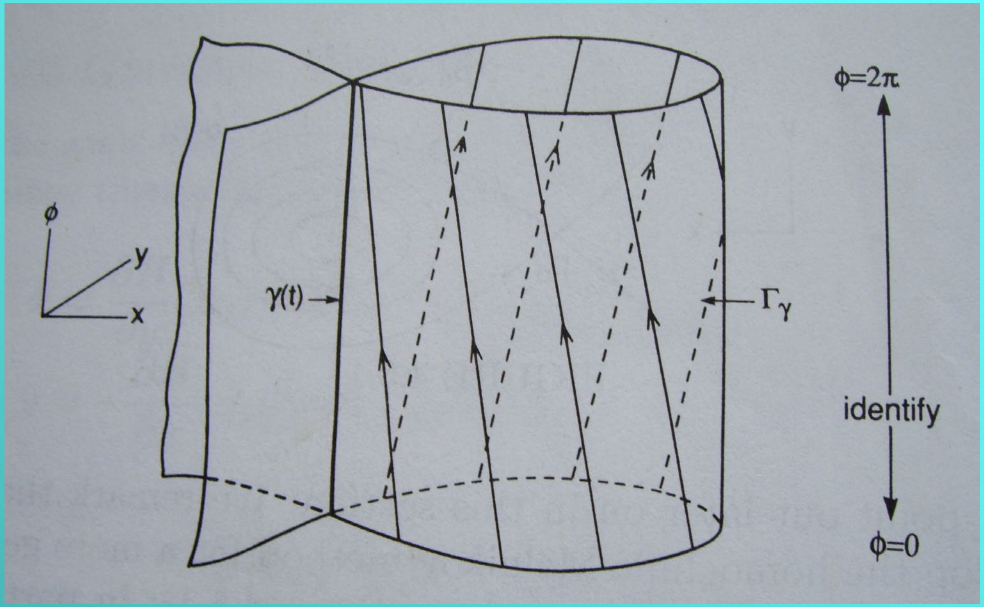
$$\dot{x}_1 = \frac{\partial H(x_i)}{\partial x_2} + \varepsilon f_1(x_i, t)$$

$$\dot{x}_2 = -\frac{\partial H(x_i)}{\partial x_1} + \varepsilon f_2(x_i, t)$$

$$f_i(x_i, t+T) = f_i(x_i, t)$$

$\gamma(t)$: The fixed point considered as hyperbolic periodic orbit

Γ_γ: The homoclinic manifold

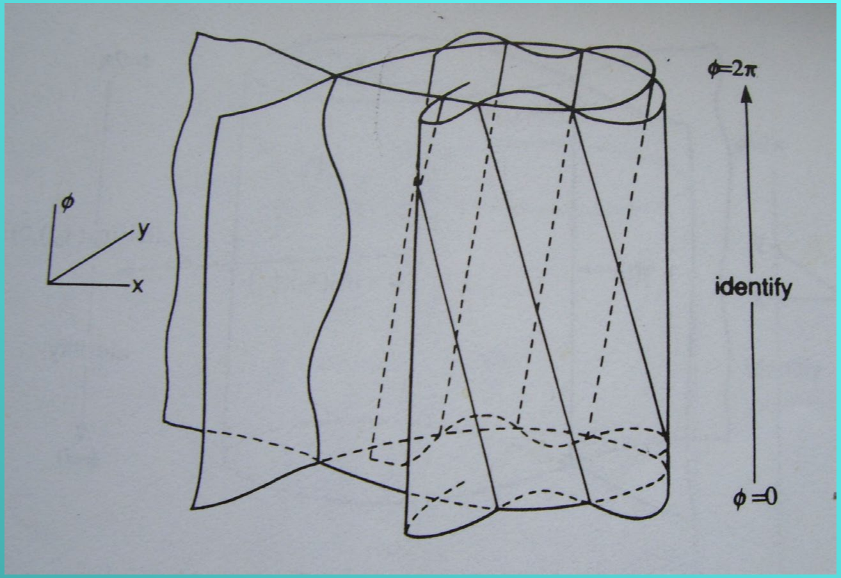


$$\varepsilon = 0$$

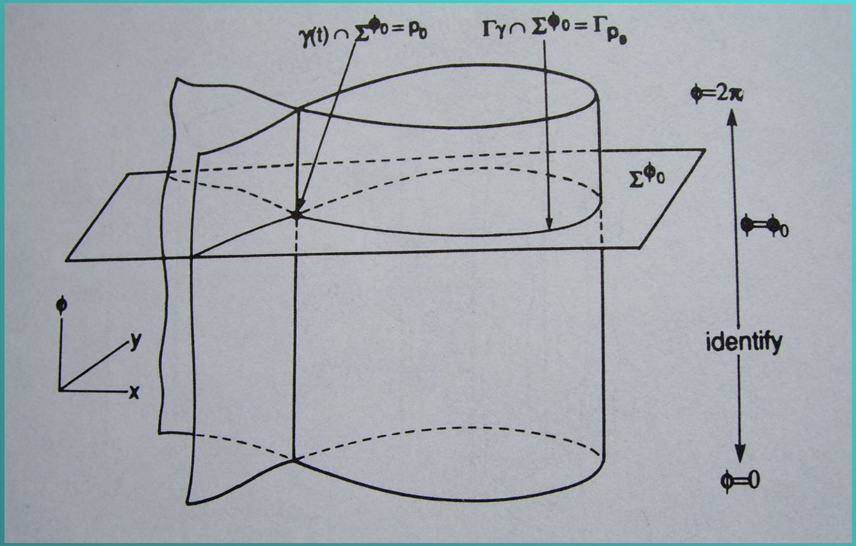
$$\dot{x}_1 = \frac{\partial H(x_i)}{\partial x_2} + \varepsilon f_1(x_i, \varphi)$$

$$\dot{x}_2 = -\frac{\partial H(x_i)}{\partial x_1} + \varepsilon f_2(x_i, \varphi)$$

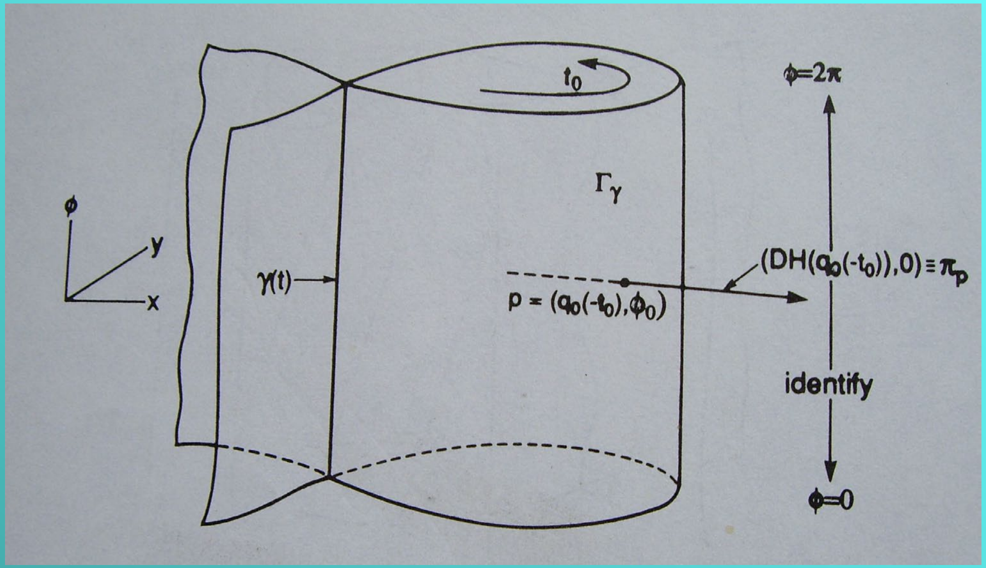
$$\dot{\varphi} = \omega \qquad \omega = \frac{2\pi}{T}$$



$\varepsilon \neq 0$



$\varepsilon = 0$



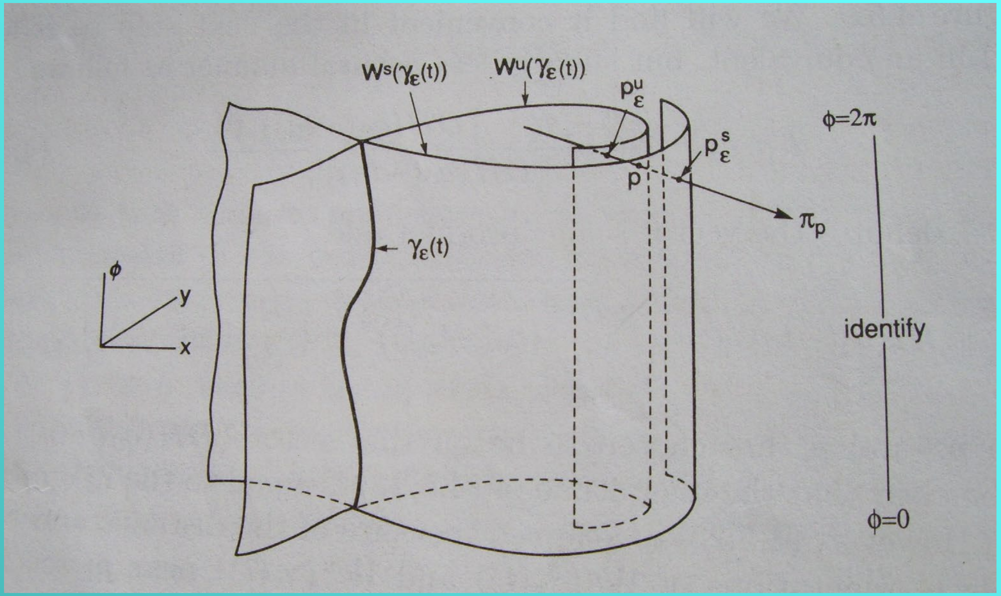
$$\pi_p = DH(q_0(-t_0),0),$$

$$q = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\gamma_\varepsilon(t)$: The perturbed hyperbolic periodic orbit

W^s: The stable manifold

W^u: The unstable manifold



$\varepsilon \neq 0$

$$d(t_0, \varphi_0, \varepsilon) = \frac{DH(q_0(-t_0)) \bullet (q^u_\varepsilon(0) - q^s_\varepsilon(0))}{\left\| DH(q_0(-t_0)) \right\|}$$

$$p^u_\varepsilon = (q^u_\varepsilon, \phi_0), \quad p^s_\varepsilon = (q^s_\varepsilon, \phi_0)$$

$$d(t_0,\varphi_0,\varepsilon)=\varepsilon\frac{DH(q_0(-t_0))\bullet(q_1^u(0)-q_1^s(0))}{\left\|DH(q_0(-t_0))\right\|}+O(\varepsilon^2)$$

The **Mel'nikov function** is

$$M(t_0,\varphi_0)=DH(q_0(-t_0))\bullet(q_1^u(0)-q_1^s(0))$$

The **time-dependent** Mel'nikov function is

$$M(t_0,\varphi_0,t)=DH(q_0(t-t_0))\bullet(q_1^u(t)-q_1^s(t))$$

$$\frac{dq_1^{u,s}}{dt} = JD^2H(q_0(t-t_0))q_1^{u,s}(t) + f(q_0(t-t_0), \varphi(t)),$$

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad DH = \begin{pmatrix} \dfrac{\partial H}{\partial x_1} \\ \dfrac{\partial H}{\partial x_2} \end{pmatrix}, \quad D^2H = \begin{pmatrix} \dfrac{\partial^2 H}{\partial x_1^2} & \dfrac{\partial^2 H}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 H}{\partial x_1 \partial x_2} & \dfrac{\partial^2 H}{\partial x_2^2} \end{pmatrix}$$

Therefore, the Mel'nikov function becomes

$$\boxed{M(t_0, \varphi_0) = \int_{-\infty}^{\infty} DH(q_0(t-t_0)) \cdot f(q_0(t-t_0), \varphi(t))\,dt}$$

The Mel'nikov Theorem

If

$$\boxed{\begin{array}{ll} 1. & M(\overline{t_0}, \overline{\varphi}_0) = 0 \\ 2. & \left.\dfrac{\partial M(t_0, \varphi_0)}{\partial t_0}\right|_{\overline{t_0}, \overline{\varphi}_0} \neq 0 \end{array}}$$

then the stable and unstable manifolds of the perturbed system **intersect transversally** for some values

$$t_0 = \overline{t_0} + O(\varepsilon), \quad \varphi_0 = \overline{\varphi}_0 + O(\varepsilon)$$

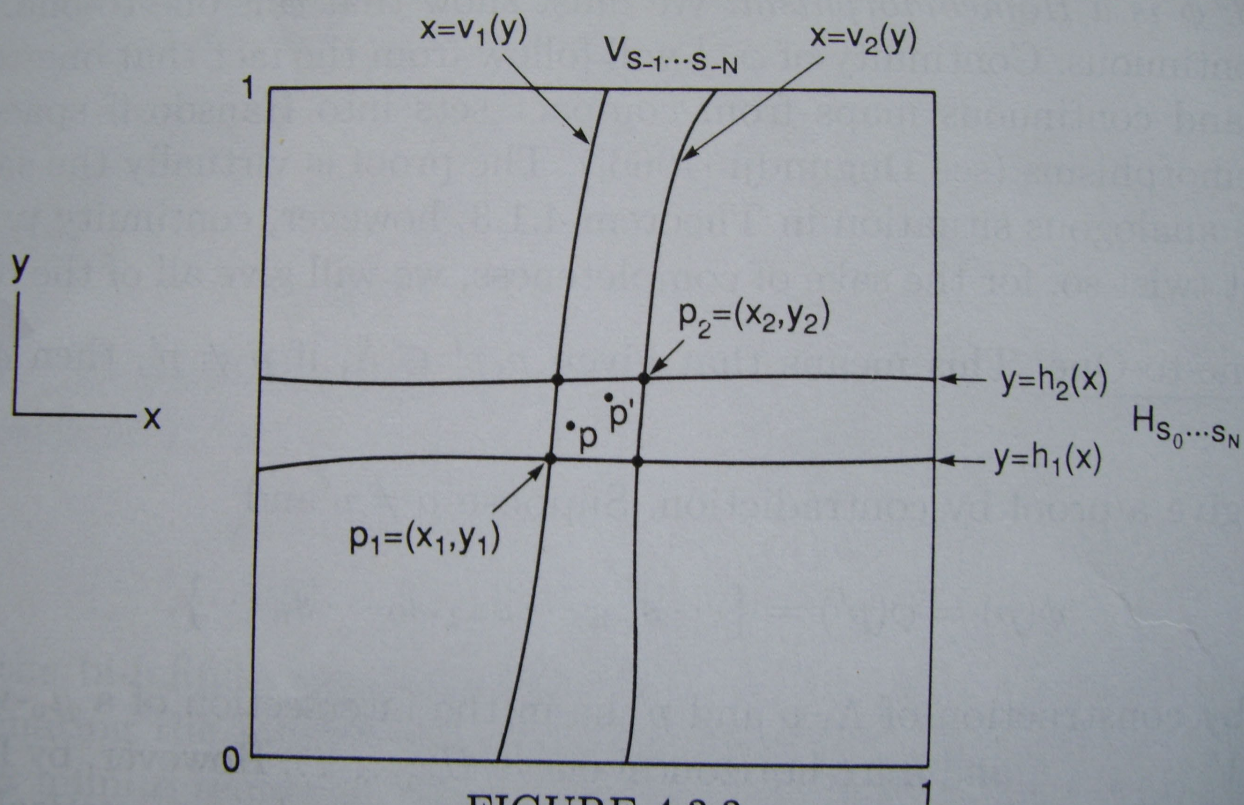
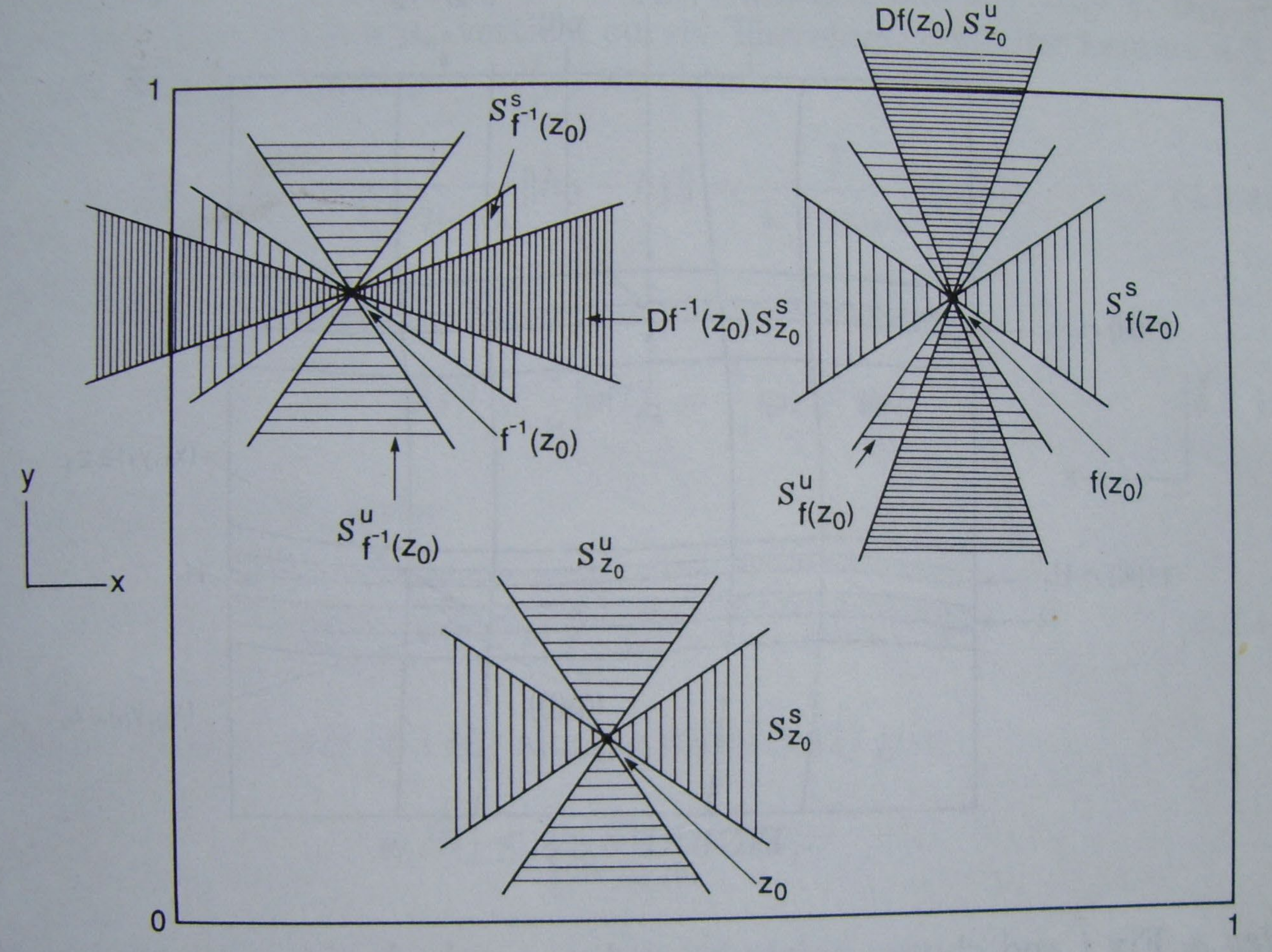


FIGURE 4.3.3.


$Df(z_0)\ S^u_{z_0}$
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$S^s_{f^{-1}(z_0)}$
$Df^{-1}(z_0)\ S^s_{z_0}$
$S^s_{f(z_0)}$
$f^{-1}(z_0)$
$S^u_{f(z_0)}$
$f(z_0)$
y
x
$S^u_{f^{-1}(z_0)}$
$S^u_{z_0}$
$S^s_{z_0}$
z_0
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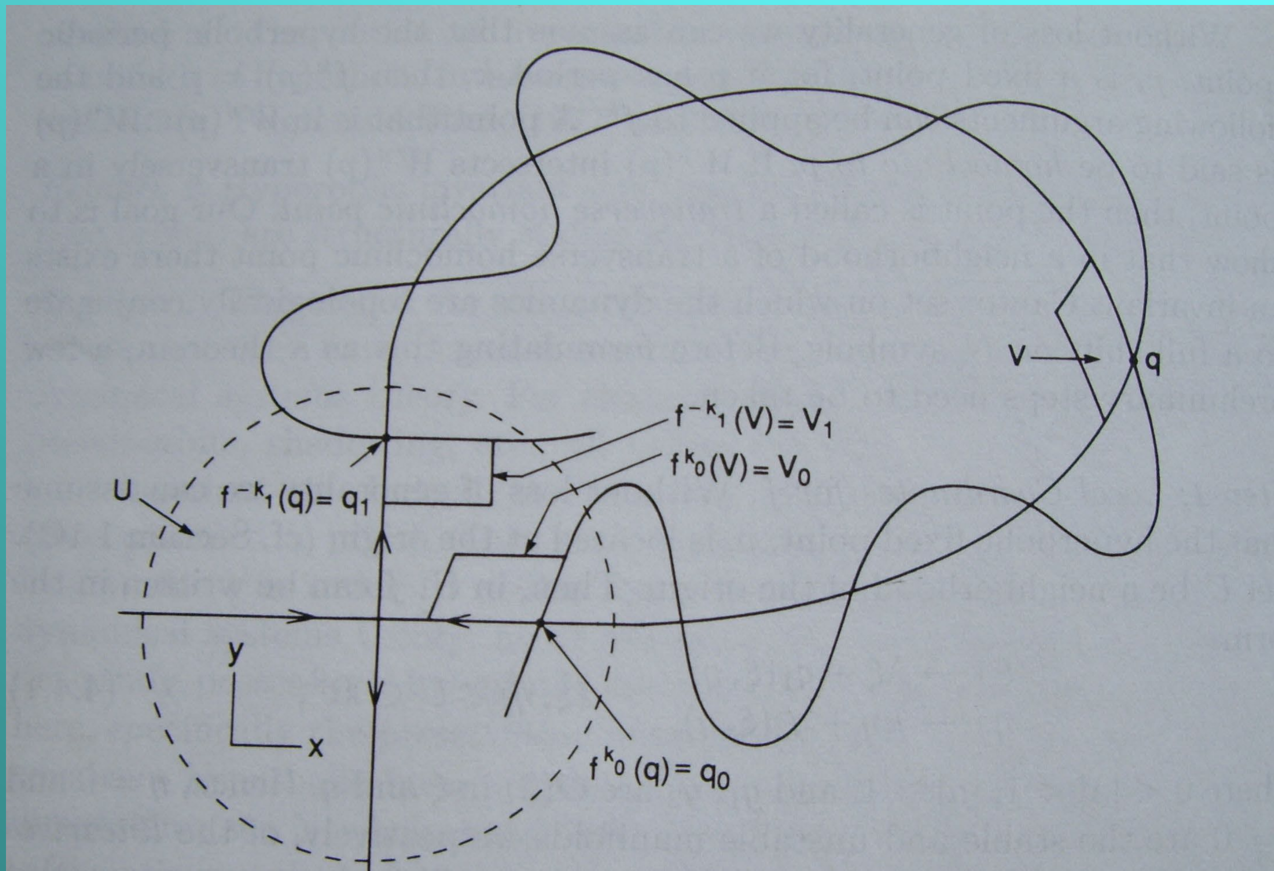
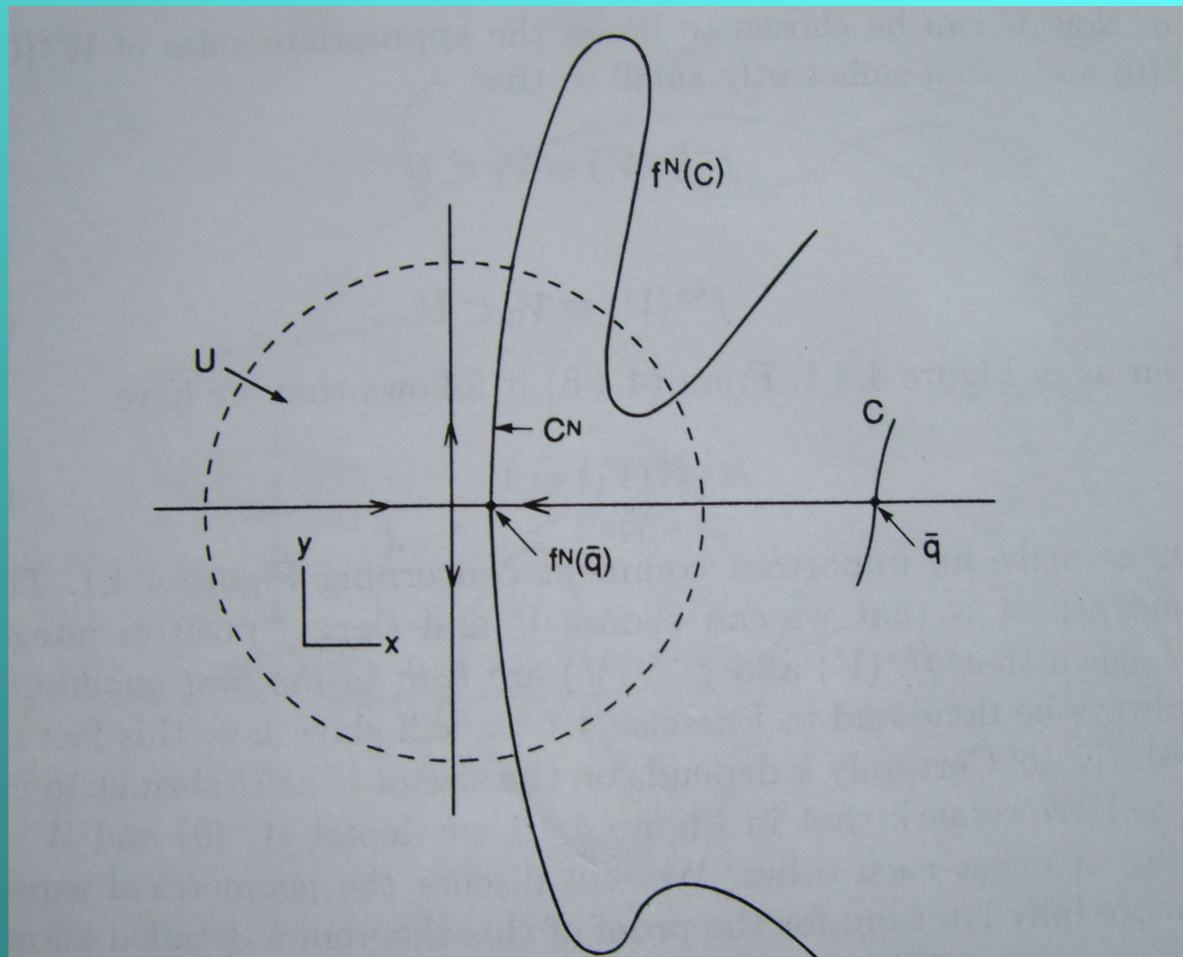
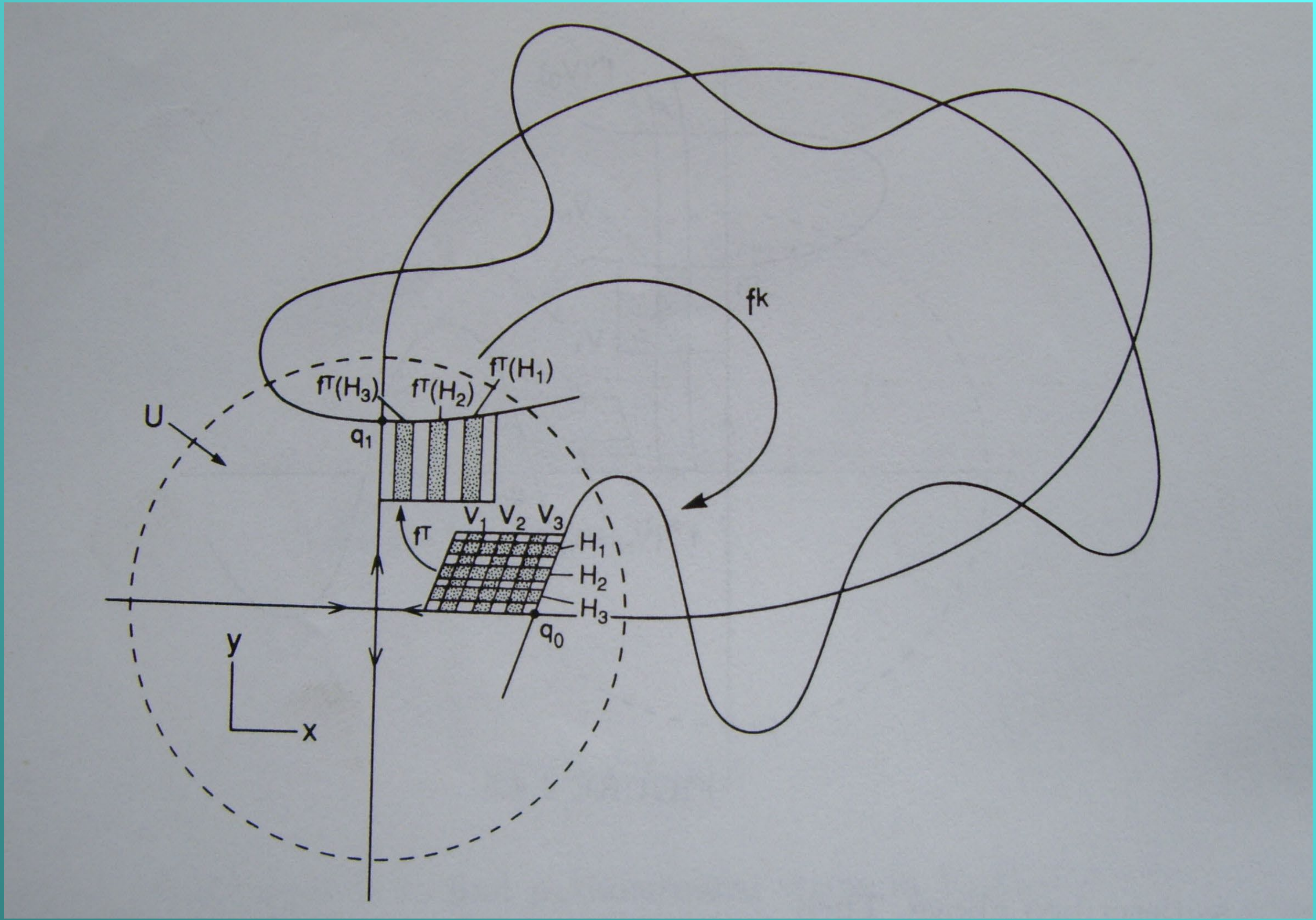
1

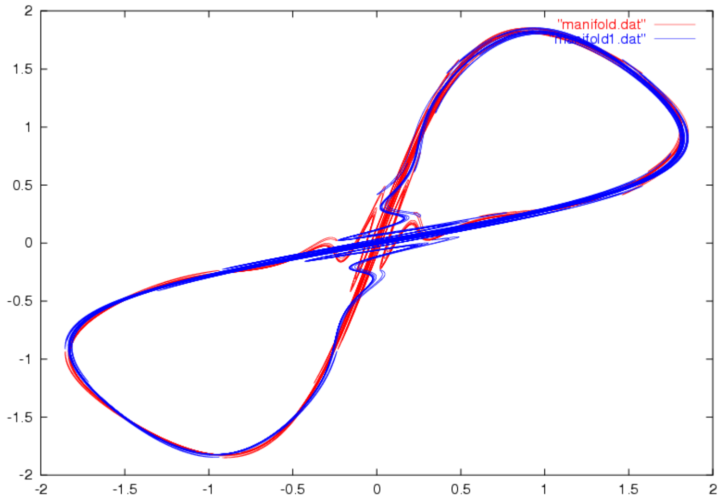

V
q
$f^{-k_1}(V) = V_1$
$f^{k_0}(V) = V_0$
U
$f^{-k_1}(q) = q_1$
y
x
$f^{k_0}(q) = q_0$



f^N(C)
U
C^N
C
f^N(q̄)
q̄
y
x



f^k
$f^T(H_3)$
$f^T(H_2)$
$f^T(H_1)$
U
q_1
f^T
V_1 V_2 V_3
H_1
H_2
H_3
q_0
y
x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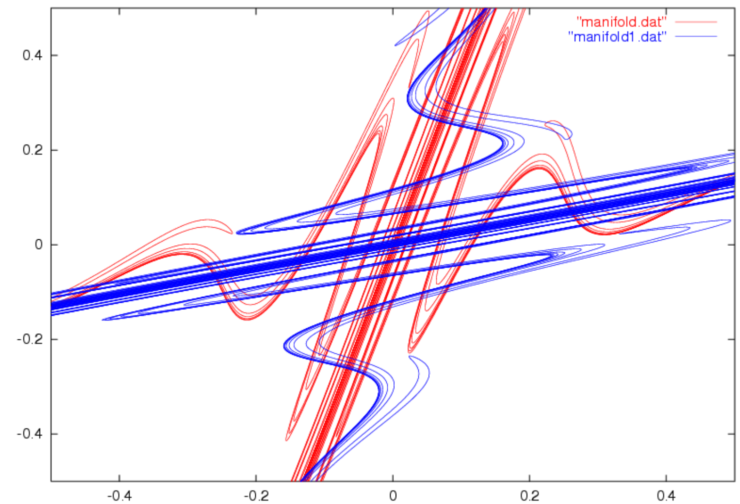
"manifold1.dat"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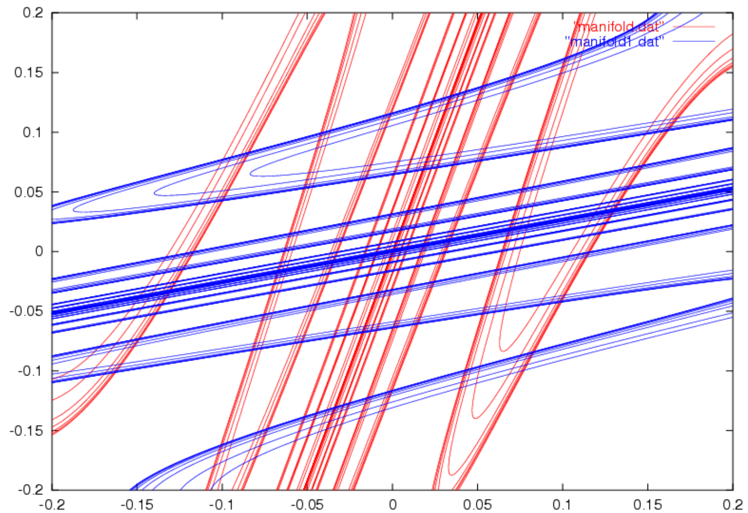
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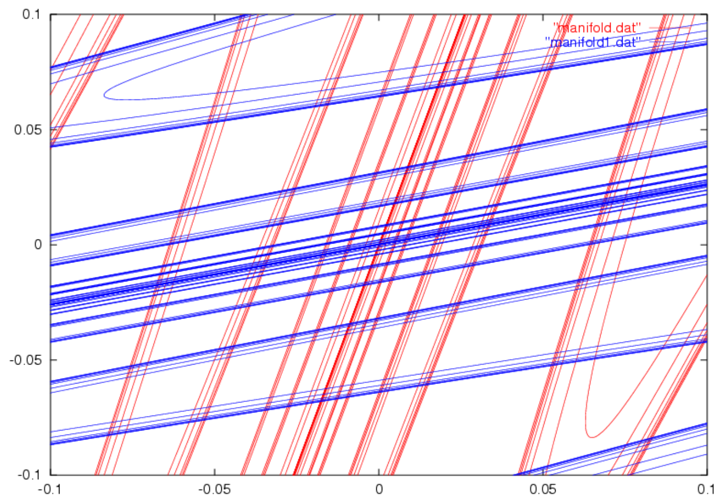
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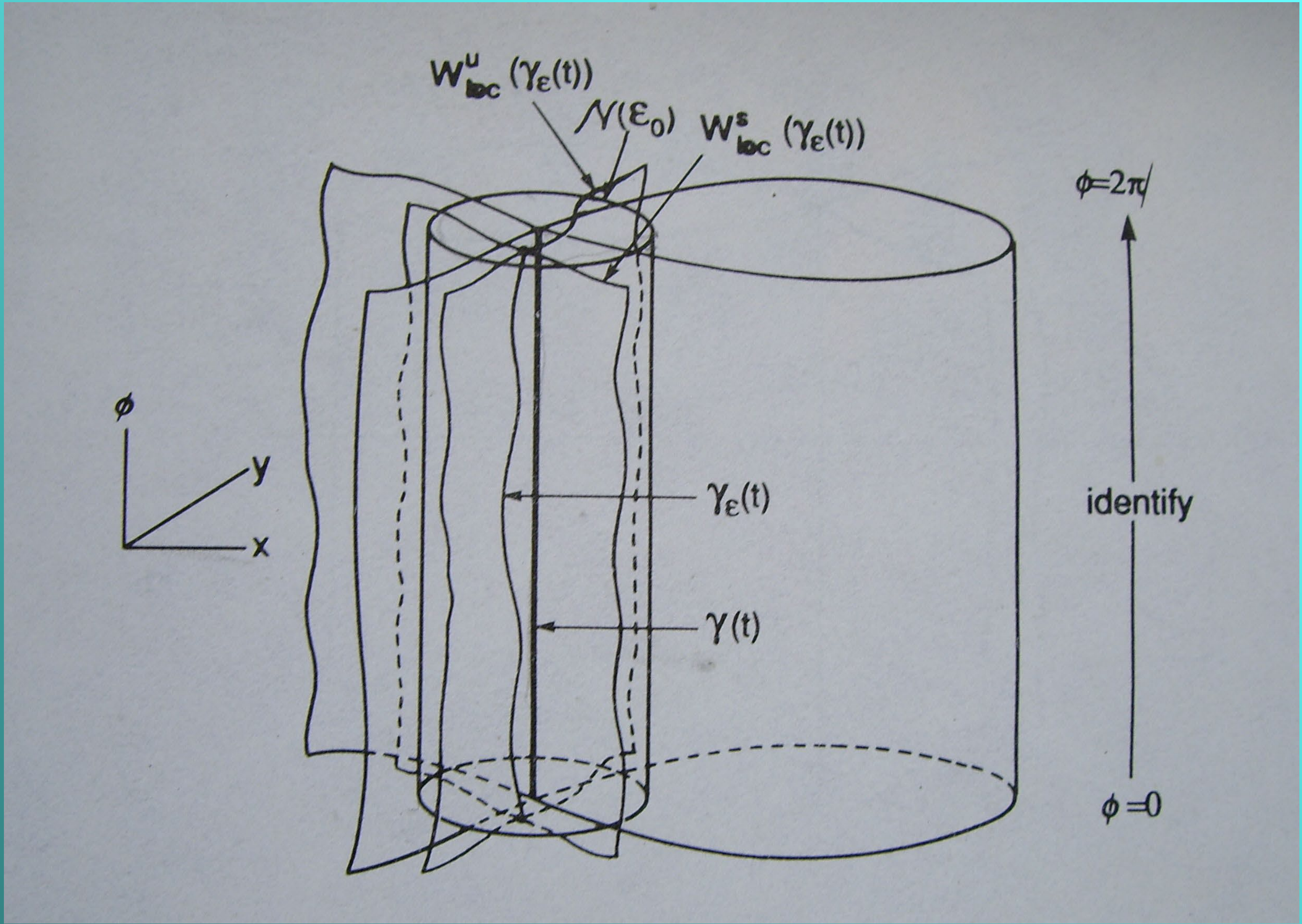
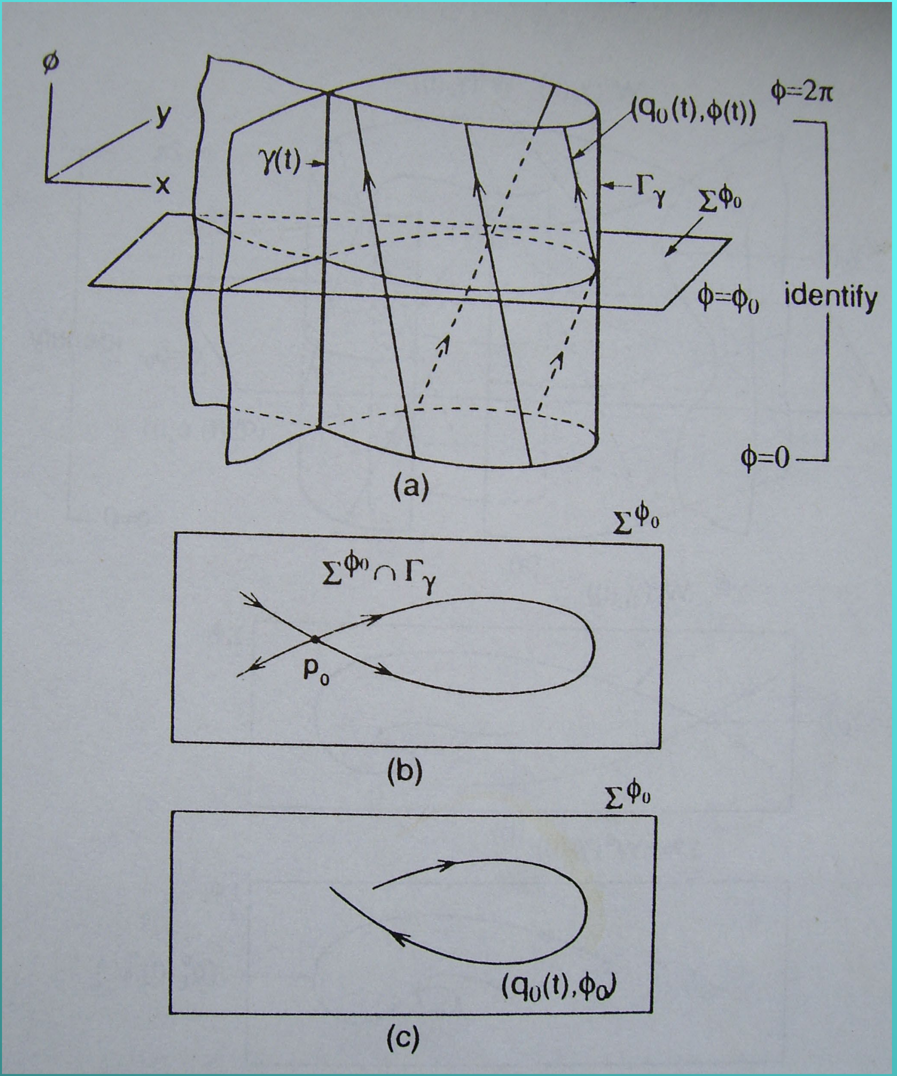
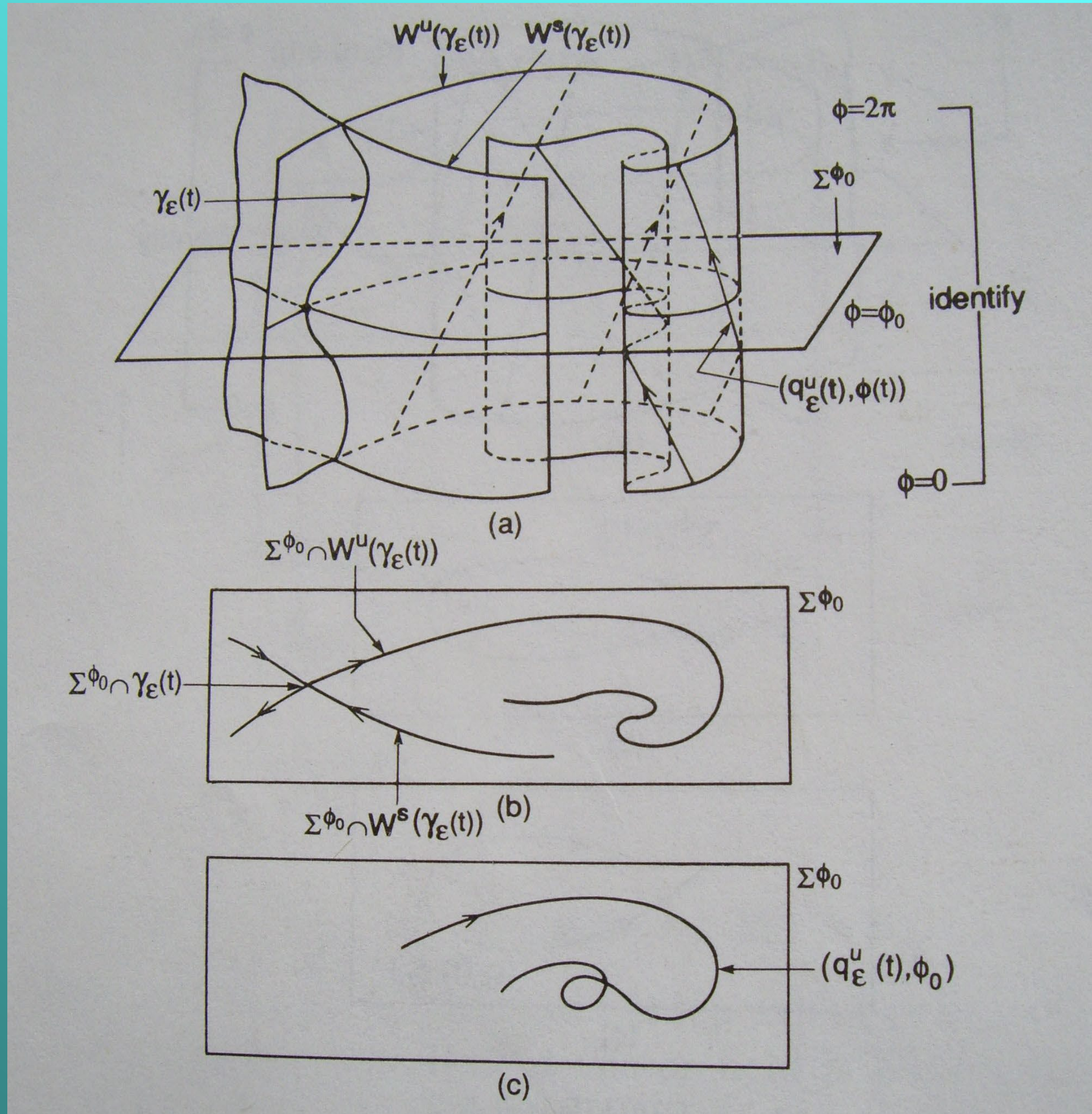


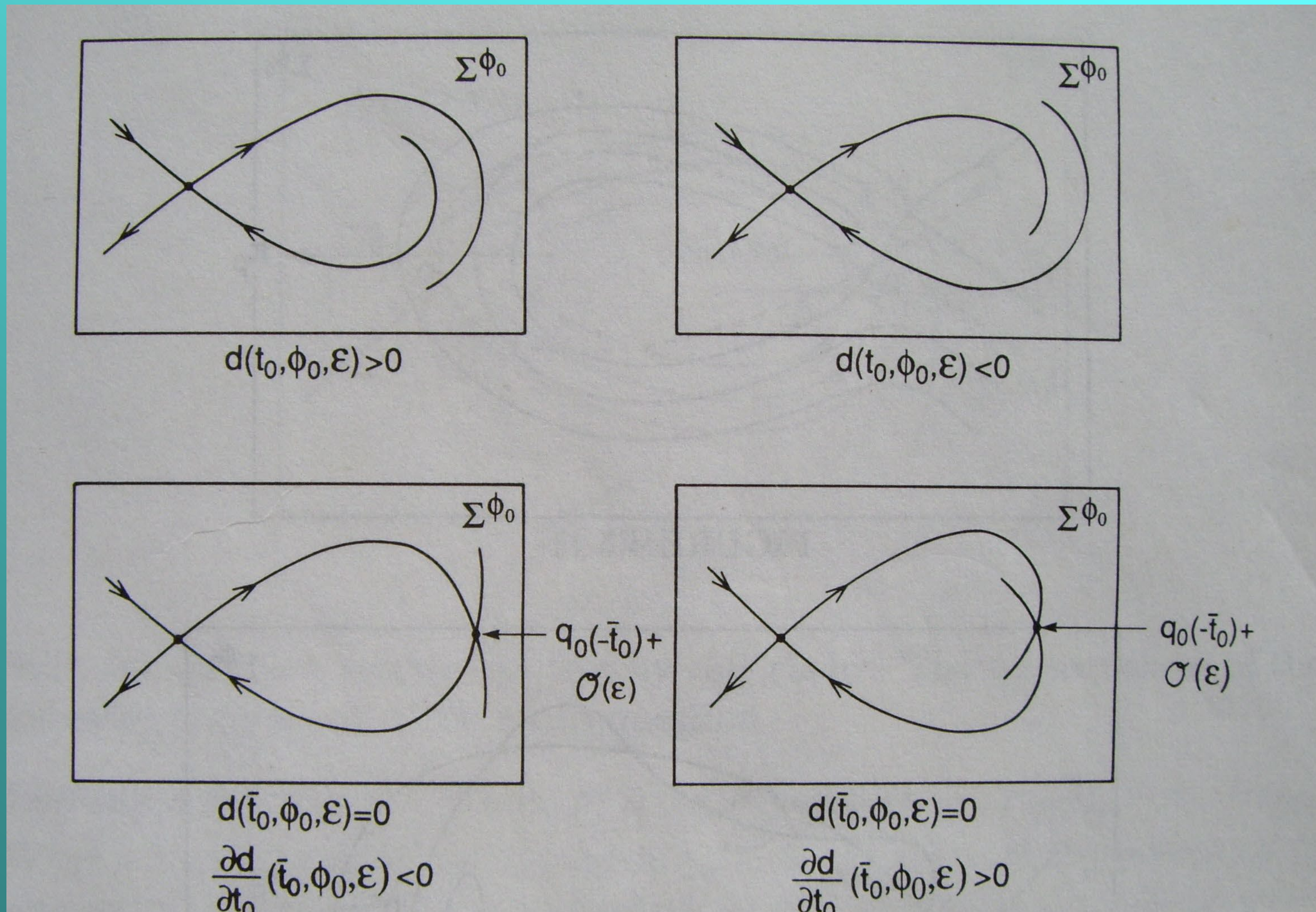
$W^u_{loc}(\gamma_\varepsilon(t))$
$\mathcal{N}(\varepsilon_0)$
$W^s_{loc}(\gamma_\varepsilon(t))$
$\phi = 2\pi$
ϕ
y
x
$\gamma_\varepsilon(t)$
identify
$\gamma(t)$
$\phi = 0$



φ
y
x
γ(t)
(q₀(t), φ(t))
Γ_γ
Σφ₀
φ=2π
φ=φ₀
identify
φ=0
(a)
Σφ₀
Σφ₀ ∩ Γ_γ
p₀
(b)
Σφ₀
(q₀(t), φ₀)
(c)



$W^u(\gamma_\varepsilon(t))$
$W^s(\gamma_\varepsilon(t))$
$\phi=2\pi$
Σ^{ϕ_0}
$\gamma_\varepsilon(t)$
identify
$\phi=\phi_0$
$(q^u_\varepsilon(t),\phi(t))$
$\phi=0$
(a)
$\Sigma^{\phi_0}\cap W^u(\gamma_\varepsilon(t))$
Σ^{ϕ_0}
$\Sigma^{\phi_0}\cap\gamma_\varepsilon(t)$
(b)
$\Sigma^{\phi_0}\cap W^s(\gamma_\varepsilon(t))$
Σ^{ϕ_0}
$(q^u_\varepsilon(t),\phi_0)$
(c)


Σ^{ϕ_0}
$d(t_0,\phi_0,\varepsilon)>0$
Σ^{ϕ_0}
$d(t_0,\phi_0,\varepsilon)<0$
Σ^{ϕ_0}
$q_0(-\bar{t}_0)+$
$\mathcal{O}(\varepsilon)$
$d(\bar{t}_0,\phi_0,\varepsilon)=0$
$\frac{\partial d}{\partial t_0}(\bar{t}_0,\phi_0,\varepsilon)<0$
Σ^{ϕ_0}
$q_0(-\bar{t}_0)+$
$\mathcal{O}(\varepsilon)$
$d(\bar{t}_0,\phi_0,\varepsilon)=0$
$\frac{\partial d}{\partial t_0}(\bar{t}_0,\phi_0,\varepsilon)>0$

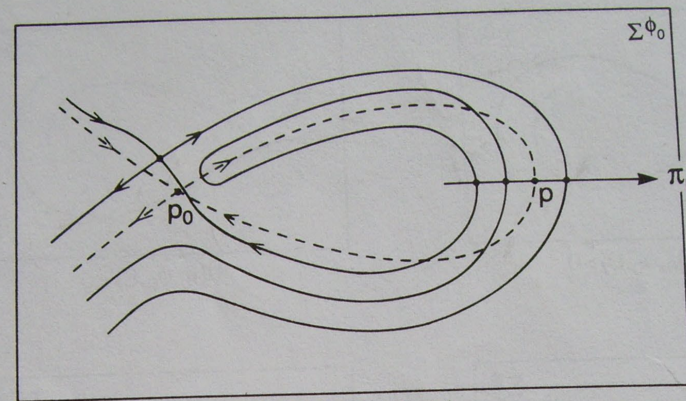


FIGURE 4.5.11.

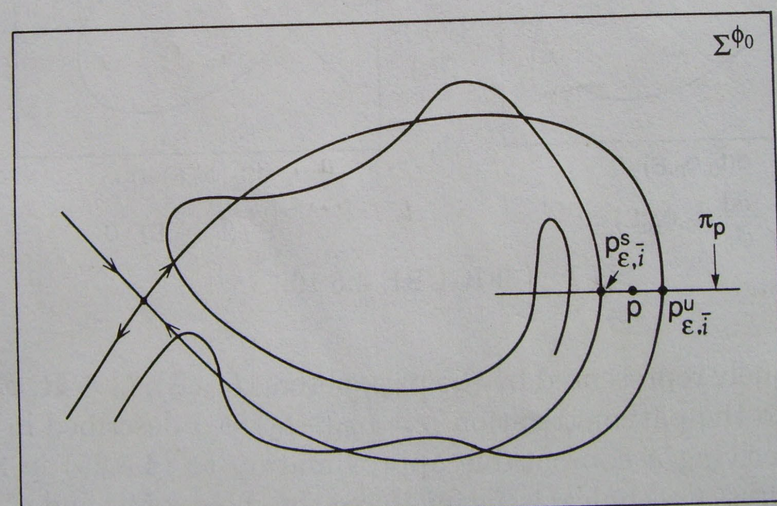
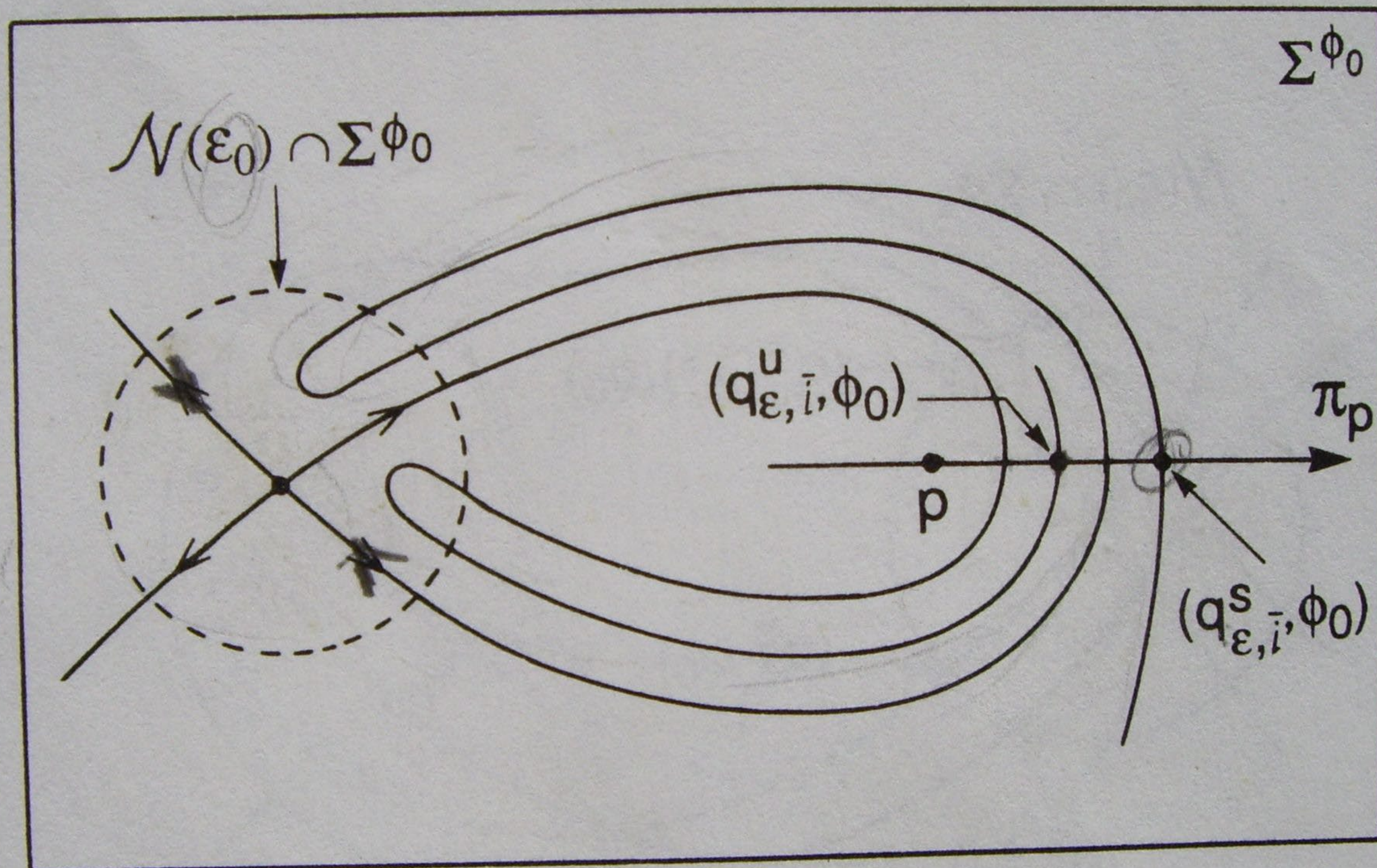
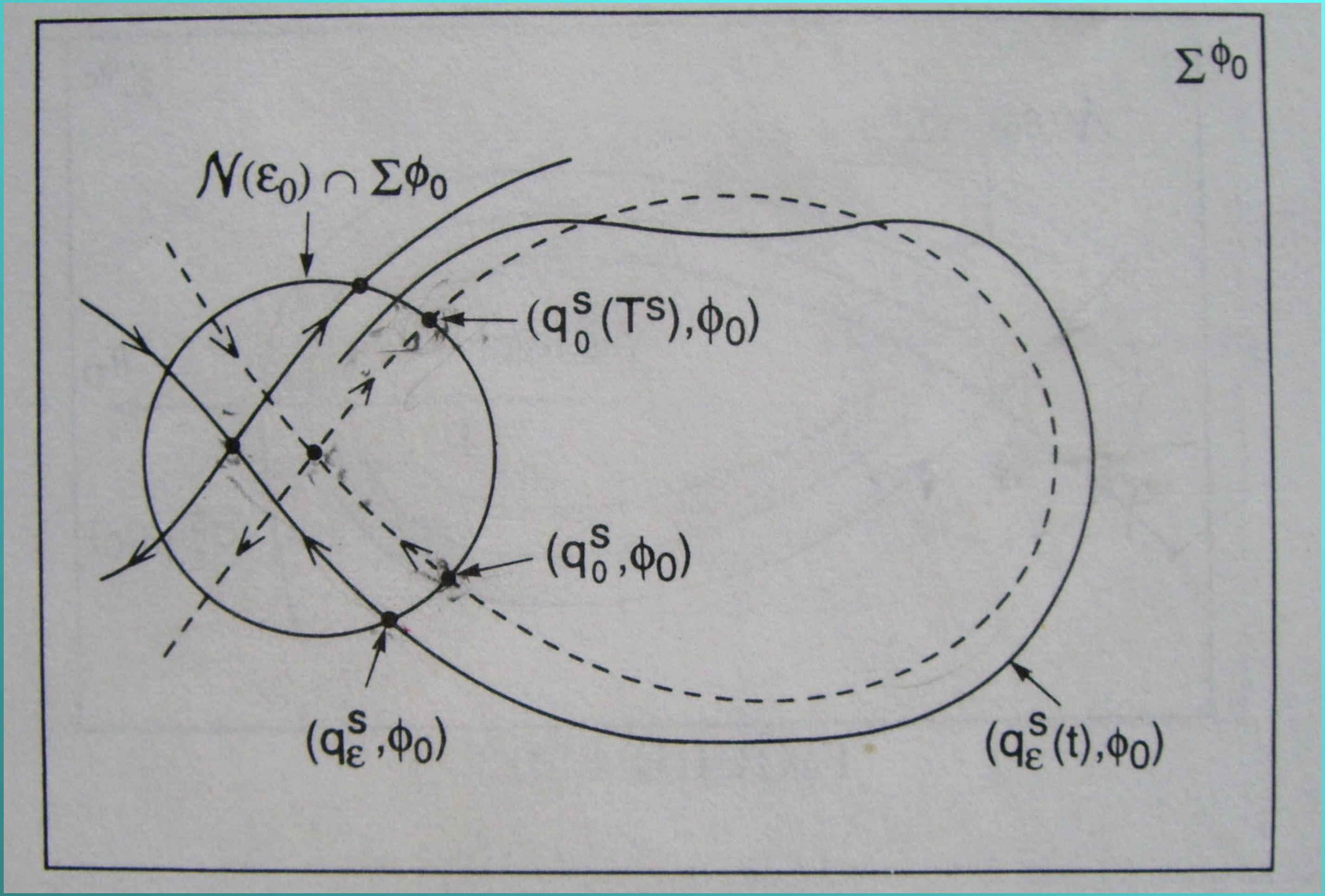
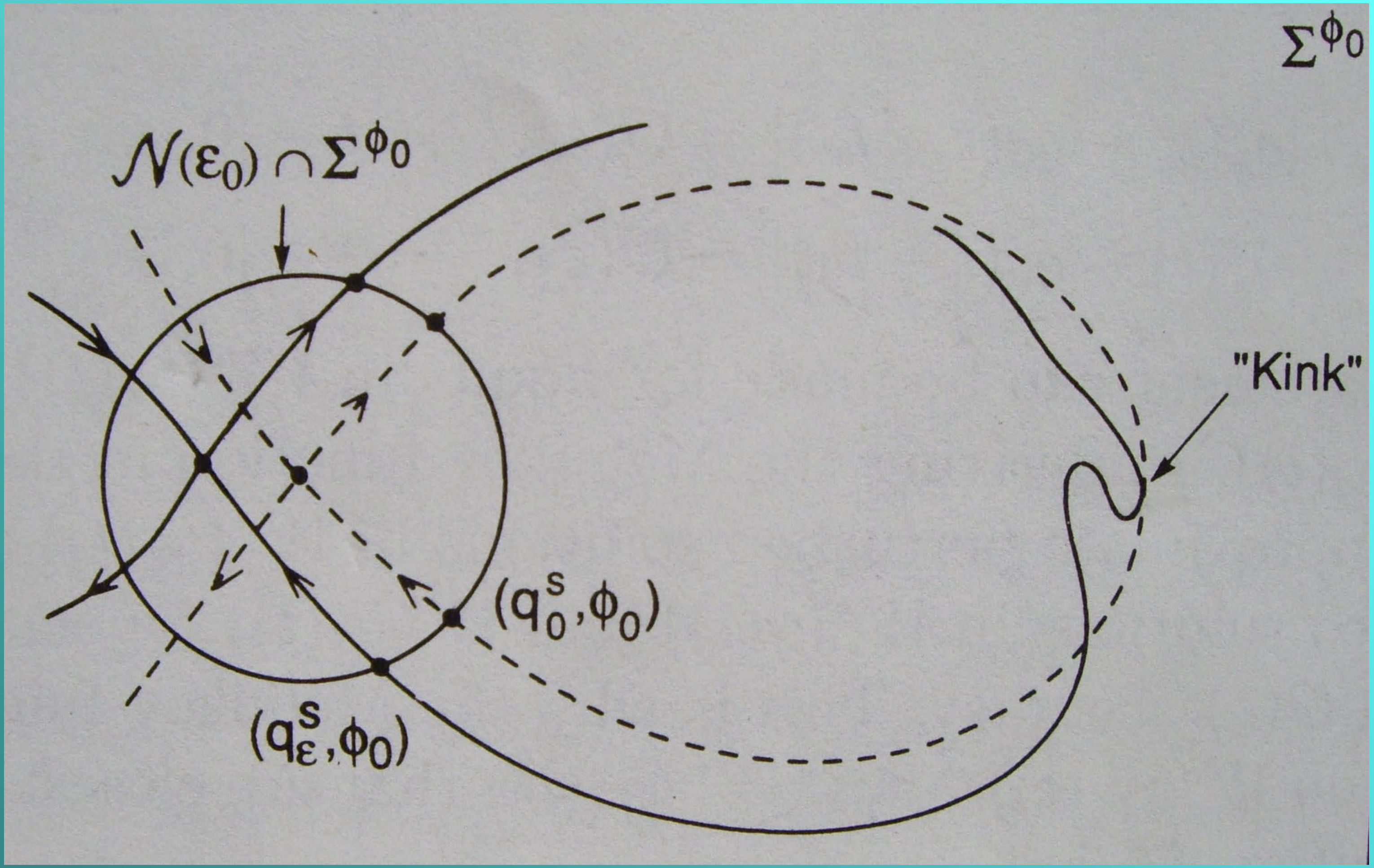


FIGURE 4.5.12.



Σ^{ϕ_0}
$\mathcal{N}(\varepsilon_0) \cap \Sigma^{\phi_0}$
$(q^u_{\varepsilon,\bar{\iota}}, \phi_0)$
π_p
p
$(q^s_{\varepsilon,\bar{\iota}}, \phi_0)$


Σ^{ϕ_0}
$\mathcal{N}(\varepsilon_0) \cap \Sigma\phi_0$
$(q_0^s(T^s),\phi_0)$
(q_0^s,ϕ_0)
(q_ε^s,ϕ_0)
$(q_\varepsilon^s(t),\phi_0)$



Σ^{ϕ_0}
$\mathcal{N}(\varepsilon_0) \cap \Sigma^{\phi_0}$
"Kink"
(q_0^s, ϕ_0)
$(q_\varepsilon^s, \phi_0)$