The Restricted Three Body Problem

and an application to the dynamical environment of 65803 Didymos

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31ο Θερινό Σχολείο – Συνέδριο ΔΥΝΑΜΙΚΑ ΣΥΣΤΗΜΑΤΑ ΚΑΙ ΠΟΛΥΠΛΟΚΟΤΗΤΑ
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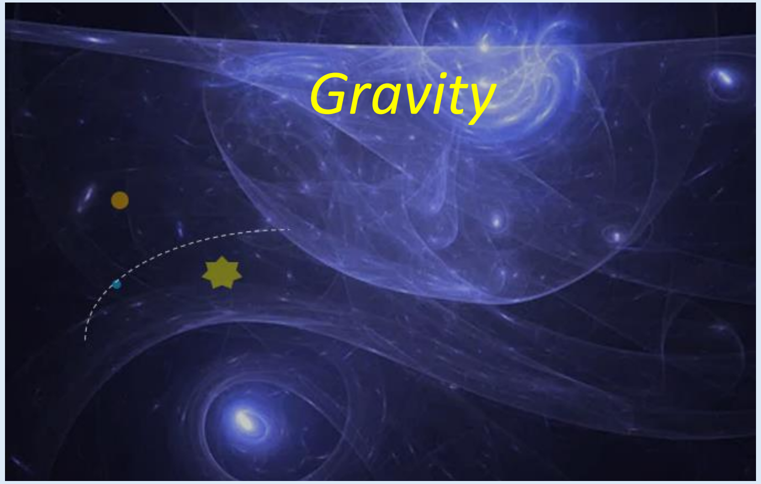
Leonhard Euler (1772)

Henri Poincare (1890)

Victor Szebehely (1967)
(*Theory of orbits*)



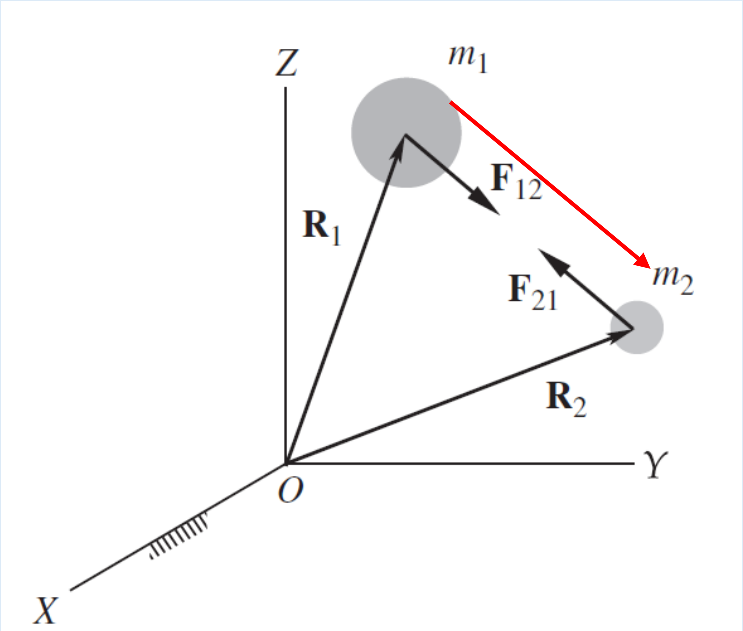
Today, is a basic model in Celestial Mechanics for :

- *Mathematical dynamical systems theory*
- *Asteroid dynamics*
- *Planetary dynamics*
- *Space mission design*

Outline

- Two body problem
- General three body problem
- The **restricted** problem
- Application to a binary asteroid system

The two body problem (2BP)



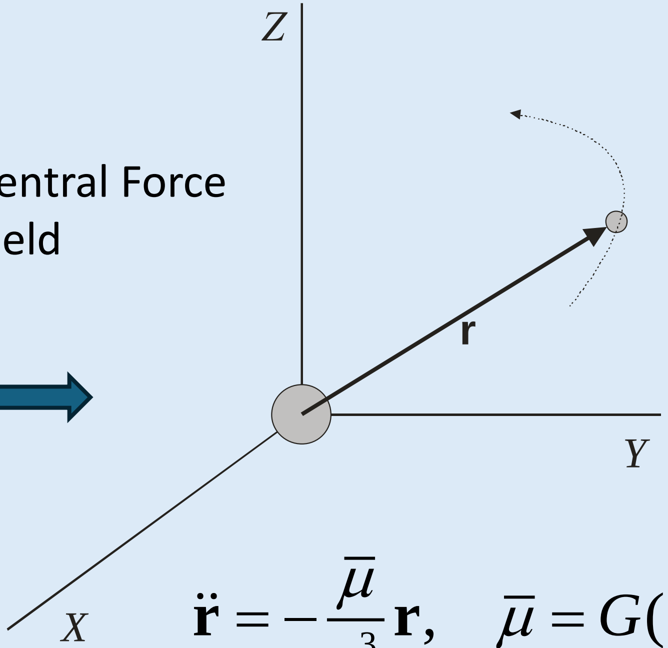
$$\mathbf{F}_{12} = -\mathbf{F}_{21} = -\frac{Gm_1m_2}{r^3}\mathbf{r}$$

$$\mathbf{r} = \mathbf{R}_2 - \mathbf{R}_1$$

$$\ddot{\mathbf{R}}_1 = -\frac{Gm_2}{r^3}\mathbf{r}$$

$$\ddot{\mathbf{R}}_2 = -\frac{Gm_1}{r^3}\mathbf{r}$$

Central Force Field



$$\ddot{\mathbf{r}} = -\frac{\bar{\mu}}{r^3}\mathbf{r}, \quad \bar{\mu} = G(m_1 + m_2)$$

$$\ddot{X} = -X/(X^2+Y^2+Z^2)^{3/2}$$

$$\ddot{Y} = -Y/(X^2+Y^2+Z^2)^{3/2}$$

$$\ddot{Z} = -Z/(X^2+Y^2+Z^2)^{3/2}$$

Hamiltonian : $H = \frac{1}{2}\mathbf{p}^2 - \frac{\bar{\mu}}{r}, \quad \mathbf{p} = \dot{\mathbf{r}}$

Integrals of motion

Angular Momentum : $\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad (L, L_Z)$

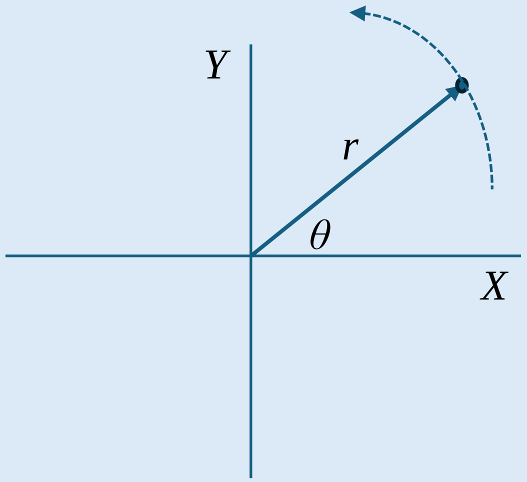
Energy : $E = \frac{1}{2}\dot{\mathbf{r}}^2 - \frac{\bar{\mu}}{r}$

Laplace-Runge-Lenz vector : $\mathbf{A} = \dot{\mathbf{r}} \times (\mathbf{r} \times \dot{\mathbf{r}}) - \bar{\mu}\frac{\mathbf{r}}{r} \quad (A, A_x)$

Super integrable

The two body problem (2BP)

L=constant $\Rightarrow$ motion on the plane **L·r**=0



$$\left.\begin{aligned}\dot{\theta} &= L/r^2 \\ \ddot{r} &= F(r) + \frac{L}{r^3}\end{aligned}\right\} \xrightarrow{r=r(\theta)} \frac{d^2u}{d\theta^2} + u = \frac{F(u)}{L^2u^2}$$

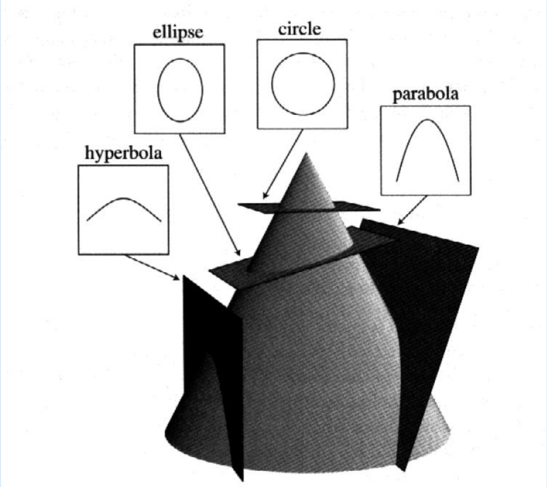
$$u = 1/r$$

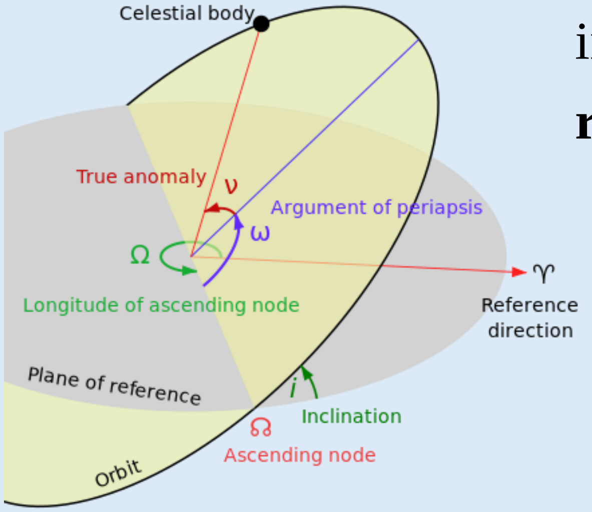
$$\xrightarrow{F(u)=\bar{\mu}u^2} \frac{d^2u}{d\theta^2} + u = A$$

$$A = \bar{\mu}/L^2 = \text{const.}$$

$$\Rightarrow \quad r = \frac{L^2/\bar{\mu}}{1 + e\cos(\theta - \omega)}$$

$$\begin{cases} e = 0 & circle \\ 0 < e < 1 & ellipse \\ e = 1 & parabola \\ e > 1 & hyperbola \end{cases}$$





initial conditions

$$\mathbf{r}(0) = \mathbf{r}_0, \quad \dot{\rho}(0) = \boldsymbol{\upsilon}_0$$

$$\downarrow$$

$$E, L$$

$$\downarrow$$

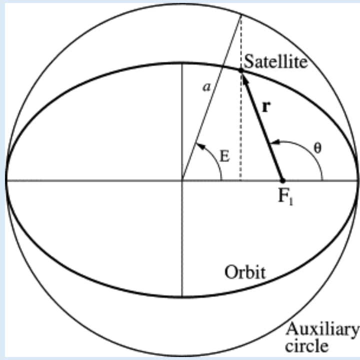
$$a, e, i, \omega, \Omega$$

$$r = r(t)?$$

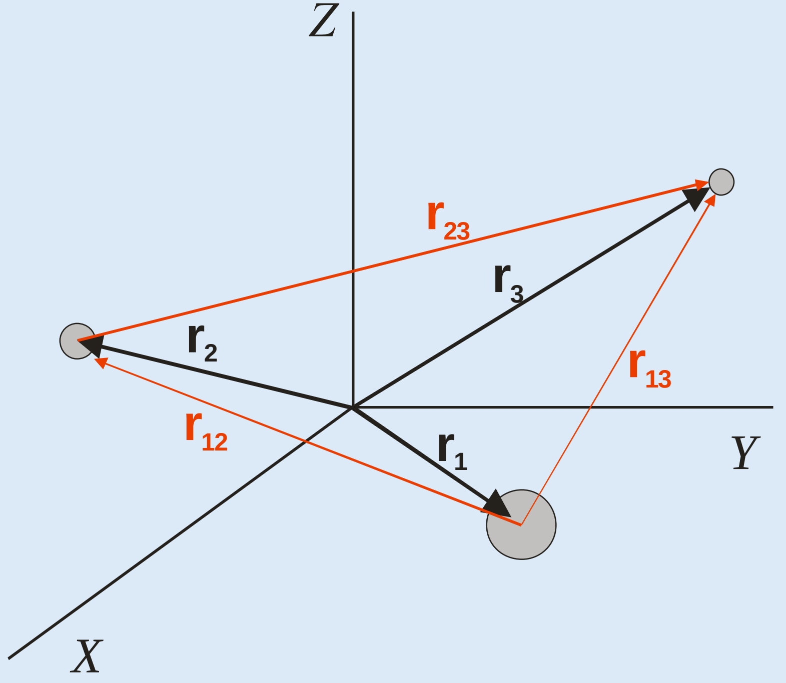
$$\boxed{r = a(1 - e\cos E)}$$

$$t = \frac{1}{n}\left(E - e\sin E\right) + \tau \quad \Rightarrow \quad E = E(t)$$

$$\tau = \text{const.}, \quad n = \frac{2\pi}{T} = 2\pi\sqrt{a^3/\mu}$$



The three body problem (3BP)



$$\ddot{\mathbf{r}}_j = -\sum_{j=1,i\neq j}^{3} \frac{Gm_i}{r_{ij}^3}\mathbf{r}_{ij}$$

$$m_1 r_1 + m_2 r_2 + m_3 r_3 = 0$$

$$m_1 \dot{r}_1 + m_2 \dot{r}_2 + m_3 \dot{r}_3 = 0$$

$$\ddot{\mathbf{r}}_2 = -\frac{Gm_1}{r_{12}^3}\mathbf{r}_{12} - \frac{Gm_3}{r_{32}^3}\mathbf{r}_{32}$$

$$\ddot{\mathbf{r}}_3 = -\frac{Gm_1}{r_{13}^3}\mathbf{r}_{13} - \frac{Gm_2}{r_{23}^3}\mathbf{r}_{23}$$

n=6 BE

Scaling of units

$$\begin{aligned} r' &\rightarrow a\,r \\ m' &\rightarrow \beta\, m \quad , \qquad \text{invariant ODEs} \Leftrightarrow \frac{\alpha^3}{\beta\gamma^2} = 1 \\ t' &\rightarrow \gamma\, t \end{aligned}$$

$$E = \frac{1}{2}\sum_{j=1}^{3} m_j \dot{r}_j^2 - \sum_{j=1,i\neq j}^{3} \frac{Gm_i m_j}{r_{ij}}$$

$$L_x = \sum_{j=1}^{3} m_j (y_j \dot{z}_j - z_j \dot{y}_j)$$

$$\mathbf{L} = \sum_{j=1}^{3} m_j r_j \times \dot{r}_j \quad \Big\} \quad L_y = \sum_{j=1}^{3} m_j (z_j \dot{x}_j - x_j \dot{z}_j)$$

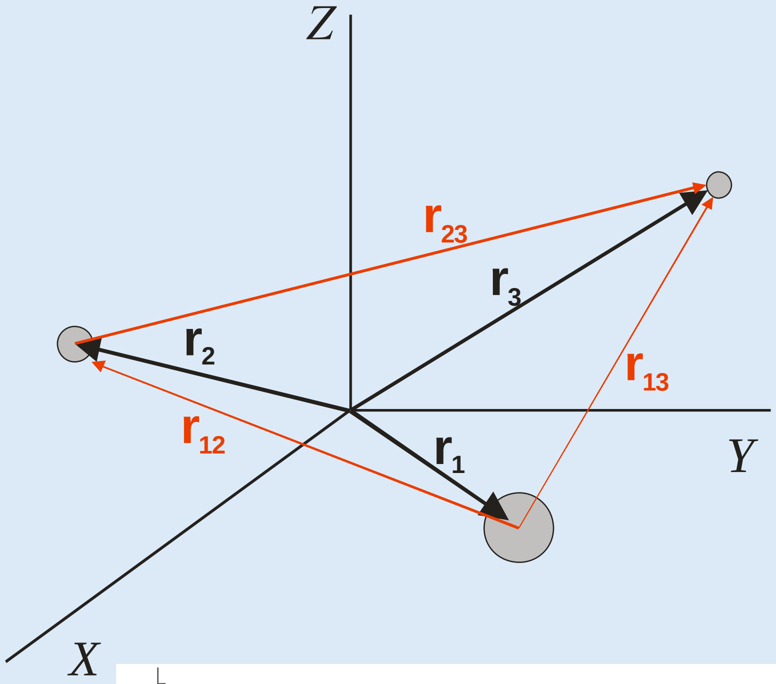
$$L_z = \sum_{j=1}^{3} m_j (x_j \dot{y}_j - y_j \dot{x}_j)$$

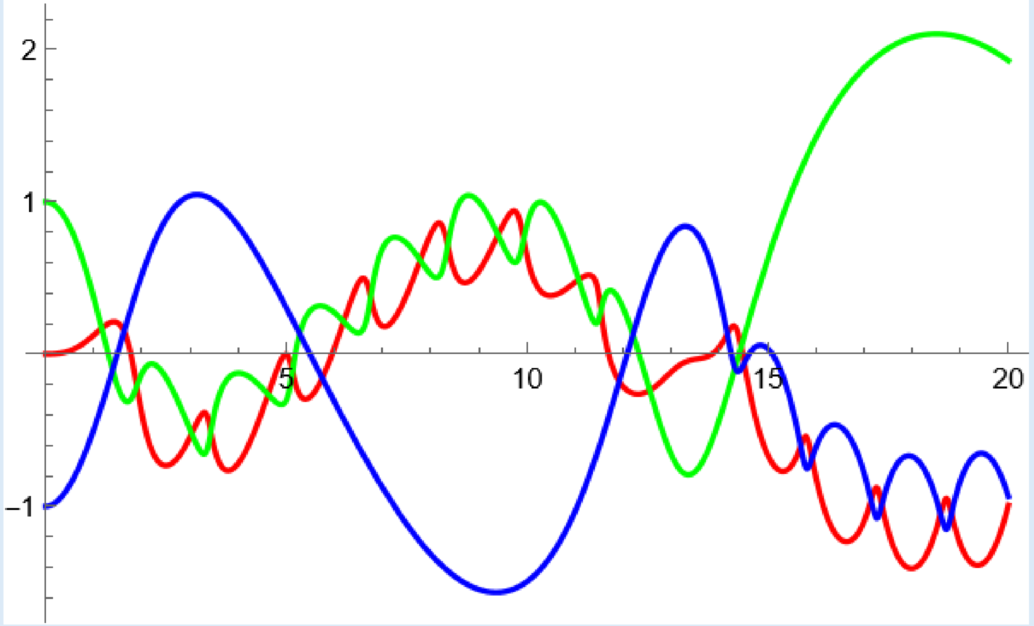
4 integrals

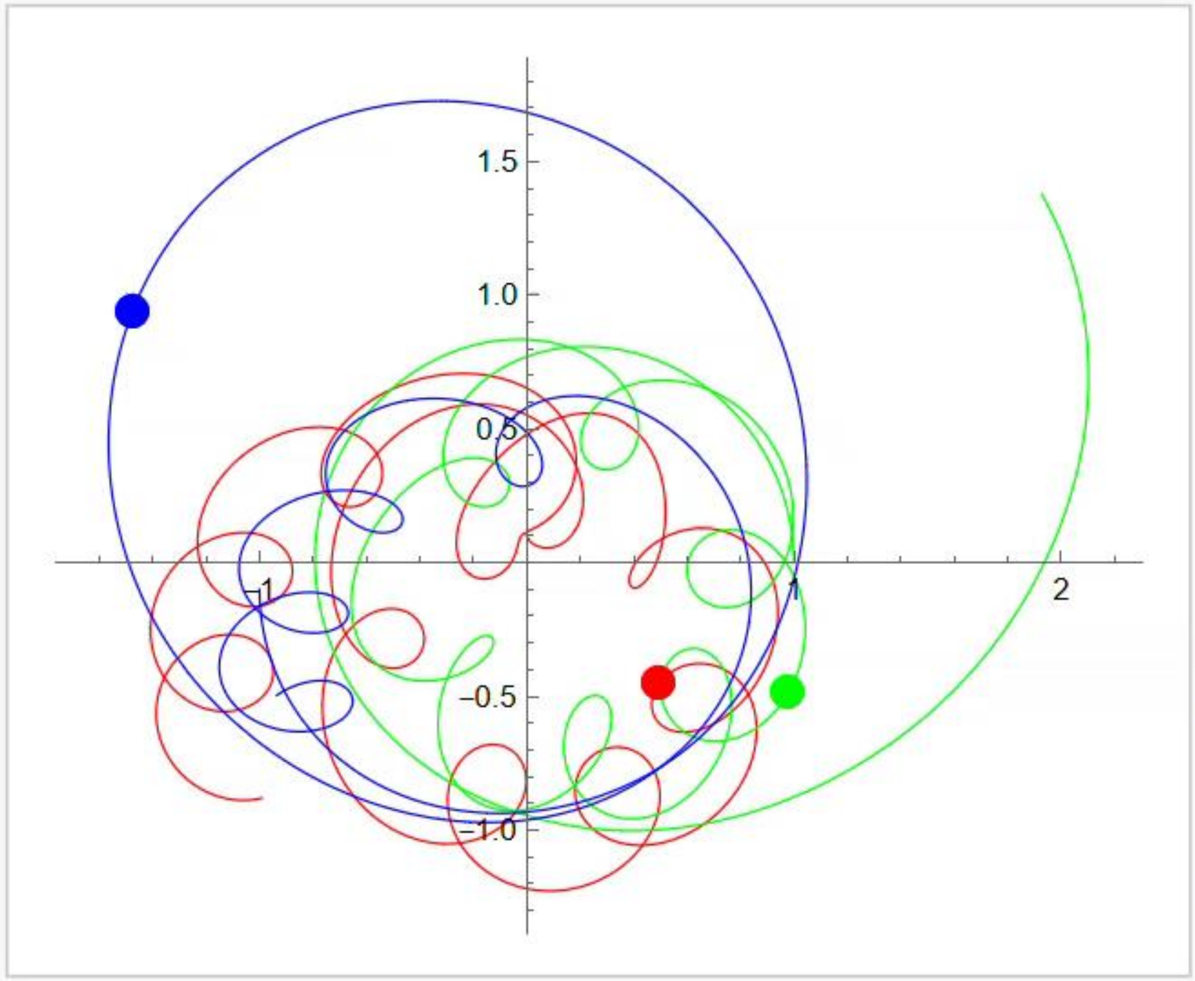
The 3BP is not integrable via Liouville's theorem

- Poincare 1890 (analytic proof)
- Henon 1966 (numerical proof)

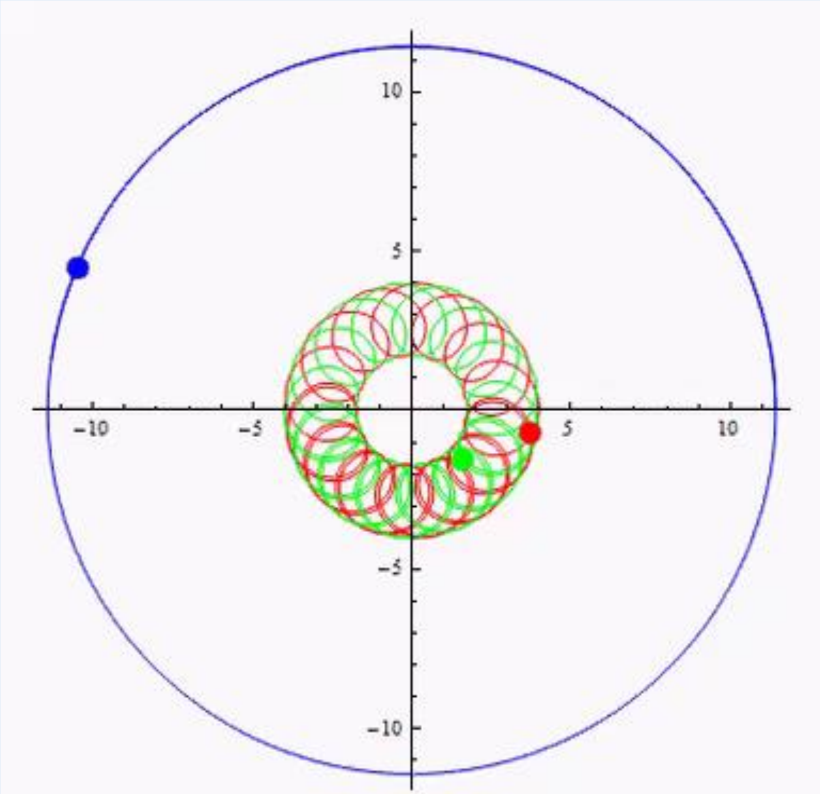
The three body problem (3BP)



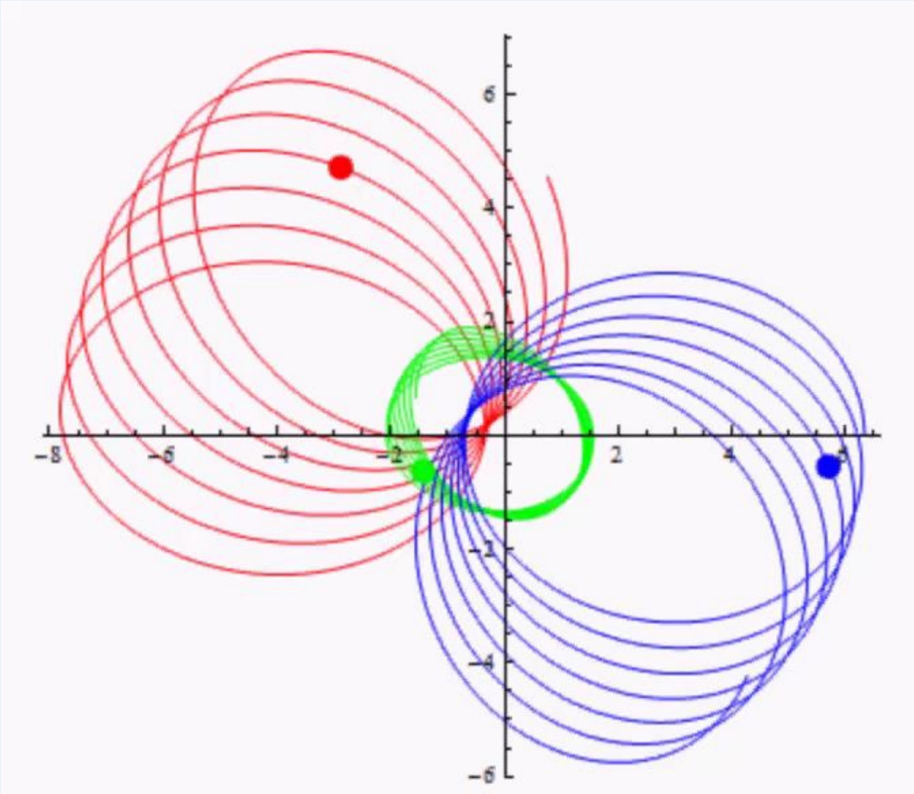




The three body problem (3BP)



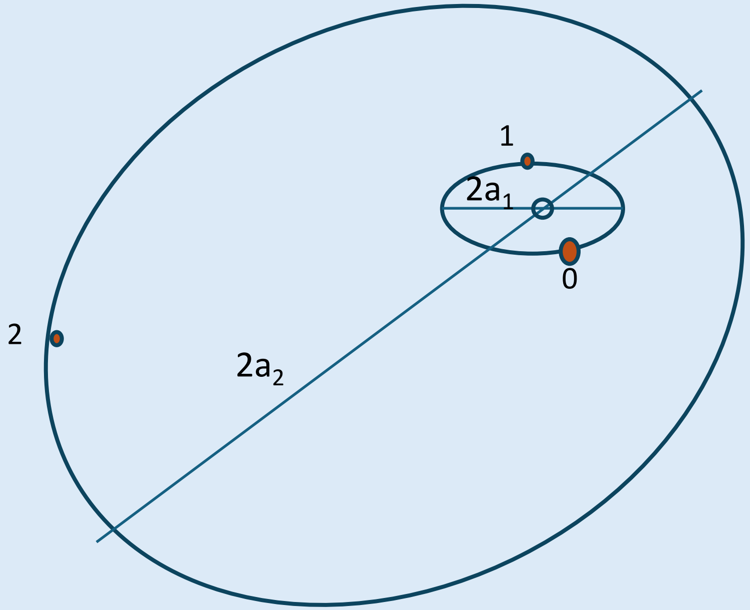
Hierarchical configuration
(stable)

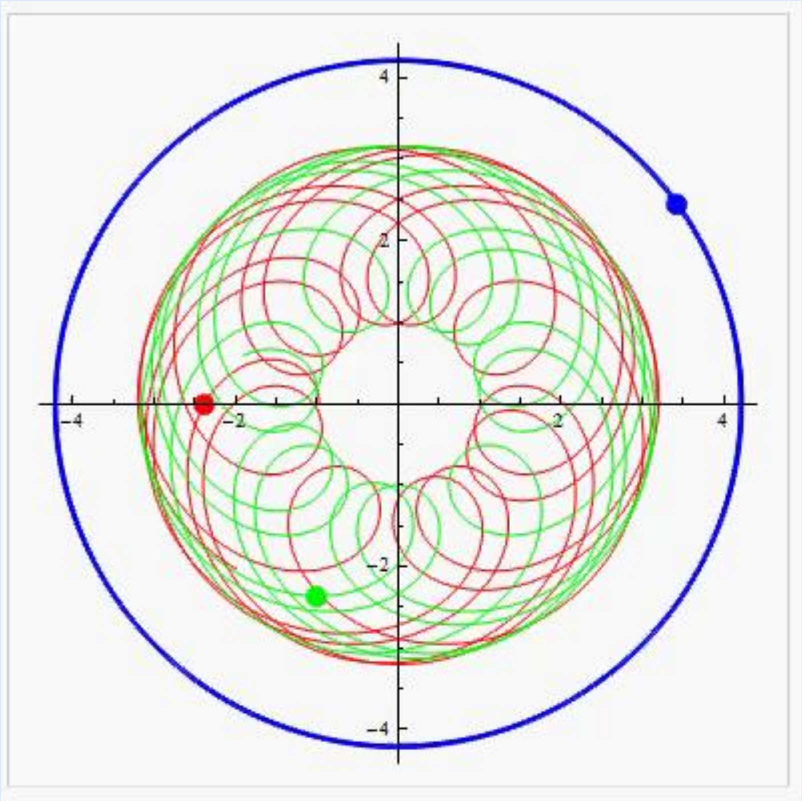


Periodic configuration
(stable)

10% of stars belong to triple systems

The three body problem (3BP)



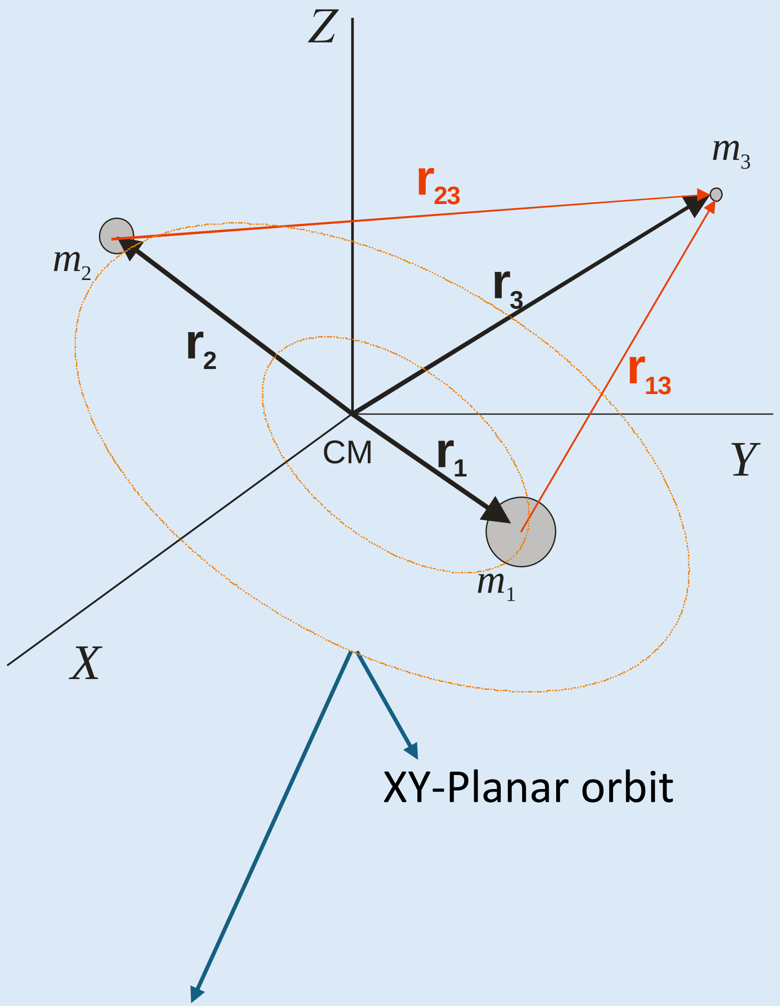


Relative orbital distance $a = a_2 / a_1$

$a_{crit} = 2.33$, $\mu = 1$

Voyatzis & Mourtetzikoglou, 2019
Frontiers in Astronomy and Space Sciences

The restricted three body problem (RTBP)



$$m_1 > m_2 \gg m_3$$

$$\ddot{\mathbf{r}}_1 = -\frac{Gm_2}{r_{12}^3}\mathbf{r}_{21}$$

$$\ddot{\mathbf{r}}_2 = -\frac{Gm_1}{r_{12}^3}\mathbf{r}_{12}$$

$$\ddot{\mathbf{r}} = -\frac{\bar{\mu}}{r^3}\mathbf{r}, \quad \bar{\mu} = G(m_1 + m_2)$$

$$\mathbf{r} = \mathbf{r}_{12} = \mathbf{r}(t)$$

$$\mathbf{r}_1 = -\frac{m_2}{m_1 + m_2}\mathbf{r}, \quad \mathbf{r}_2 = \frac{m_1}{m_1 + m_2}\mathbf{r}$$

$$\ddot{\mathbf{r}}_3 = -\frac{Gm_1}{r_{13}^3}\mathbf{r}_{13} - \frac{Gm_2}{r_{23}^3}\mathbf{r}_{23}$$

$$\mathbf{r}_{13} = (X_3 - X_1)\mathbf{i} + (Y_3 - Y_1)\mathbf{j} + Z_3\mathbf{k}$$

$$\mathbf{r}_{23} = (X_3 - X_2)\mathbf{i} + (Y_3 - Y_2)\mathbf{j} + Z_3\mathbf{k}$$

Non autonomous system of 3 DoF

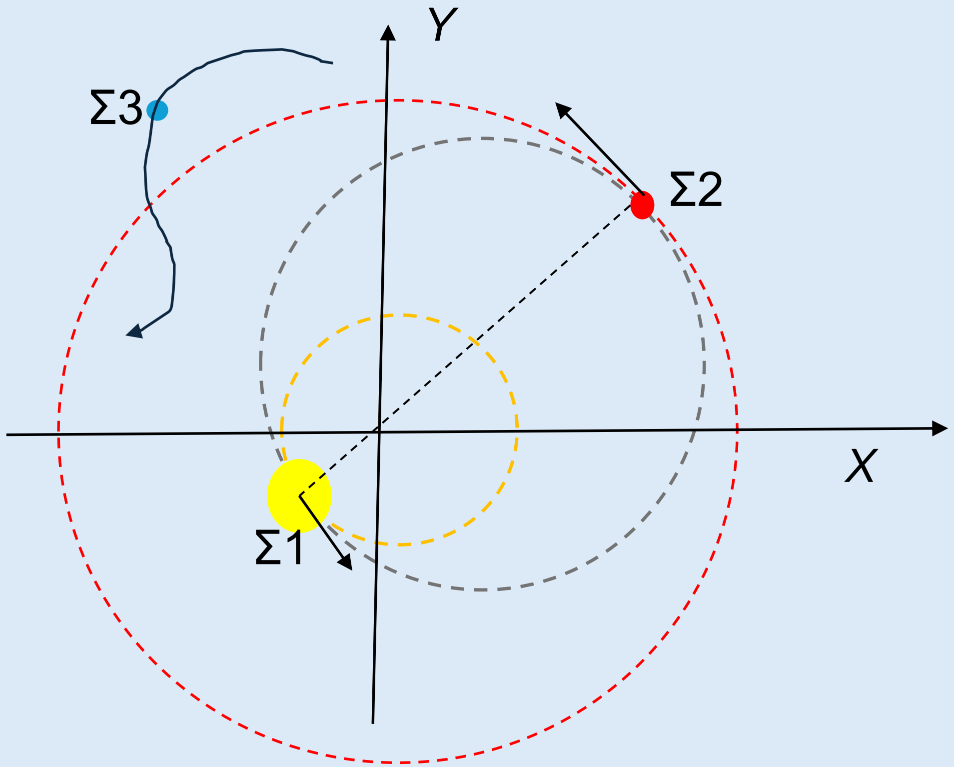
(no integrals)

- Elliptic orbit : Elliptic Restricted three body problem (ERTBP)
- Circular orbit : **Circular Restricted three body problem** (CRTBP)

Physical configurations

- Earth-Moon-Spacecraft
- Sun-Jupiter-asteroid
- Planet in a double star system

The Circular Restricted Three Body Problem (CRTBP)



$$R_1 = \frac{m_2}{m_1 + m_2} r \quad , \quad r = \text{const.}, \quad \mathbf{r} = r(\cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j})$$

$$R_2 = \frac{m_1}{m_1 + m_2} r$$

$$\ddot{X} = Gm_1 \frac{X_1 - X}{r_{13}^3} + Gm_2 \frac{X_2 - X}{r_{23}^3}$$

$$\ddot{Y} = Gm_1 \frac{Y_1 - Y}{r_{13}^3} + Gm_2 \frac{Y_2 - Y}{r_{23}^3}$$

$$\ddot{Z} = -Gm_1 \frac{Z}{r_{13}^3} - Gm_2 \frac{Z}{r_{23}^3}$$

$$r_{13} = \sqrt{(X - X_1)^2 + (Y - Y_1)^2 + Z^2},$$

$$r_{23} = \sqrt{(X - X_2)^2 + (Y - Y_2)^2 + Z^2}$$

Scaling (normalization of units)

$$\left.\begin{aligned} G(m_1 + m_2) &= 1 \\ r &= 1 \end{aligned}\right\} \quad \omega = 1$$

$$Gm_2 = \mu$$

$$Gm_1 = 1 - \mu$$

$$R_1 = \mu$$

$$R_2 = 1 - \mu$$

$$\ddot{X} = -(1-\mu)\frac{X - X_1}{r_{13}^3} - \mu\frac{X - X_2}{r_{23}^3}$$

$$\ddot{Y} = -(1-\mu)\frac{Y - Y_1}{r_{13}^3} - \mu\frac{Y - Y_2}{r_{23}^3}$$

$$\ddot{Z} = -(1-\mu)\frac{Z}{r_{13}^3} - \mu\frac{Z}{r_{23}^3}$$

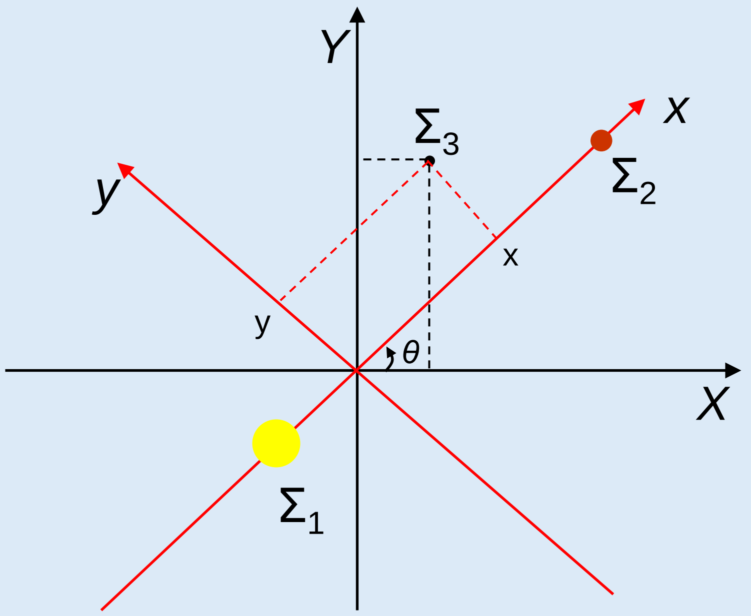
$$X_1 = -\mu\cos t$$

$$Y_1 = -\mu\sin t$$

$$X_2 = (1-\mu)\cos t$$

$$Y_2 = (1-\mu)\sin t$$

The Circular Restricted Three Body Problem (CRTBP)



Inertial OXYZ $\rightarrow$ Rotating Oxyz $\left(\dot{\theta}=\omega=1\right)$

$$X = x\cos t - y\sin t \qquad \dot{X} = \left(\dot{x}-y\right)\cdot\cos t - \left(\dot{y}+x\right)\cdot\sin t \qquad x_1 = -\mu, \quad y_1 = z_1 = 0$$

$$Y = x\sin t + y\cos t \qquad \dot{Y} = \left(\dot{y}+x\right)\cdot\cos t + \left(\dot{x}-y\right)\cdot\sin t \qquad x_2 = 1-\mu, \quad y_2 = z_2 = 0$$

$$Z = z \qquad \dot{Z} = \dot{z}$$

$$L = T - V = \frac{1}{2}(\dot{X}^2+\dot{Y}^2+\dot{Z}^2)+V$$

$$= \frac{1}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+(x\dot{y}-\dot{x}y)+\frac{1}{2}\left(x^2+y^2\right)+\frac{1-\mu}{r_1}+\frac{\mu}{r_2}$$

$$r_1 = \sqrt{(x+\mu)^2+y^2+z^2}, \qquad r_2 = \sqrt{(x-1+\mu)^2+y^2+z^2}$$

$$H = \frac{1}{2}(p_x^2+p_y^2)+p_x y - x p_y - \frac{1-\mu}{r_1}+\frac{\mu}{r_2} \quad \equiv h$$

$$p_x = \dot{x}-y, \quad p_y = \dot{y}+x$$

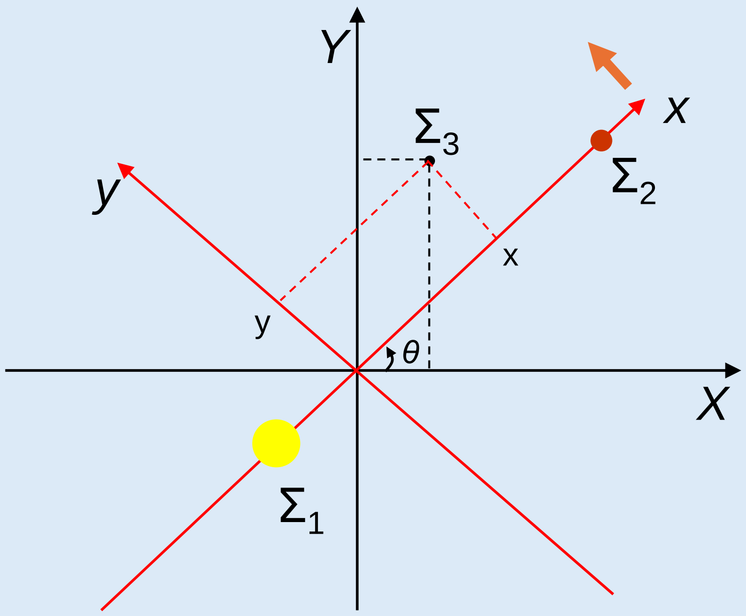
$$\ddot{x}-2\dot{y}-x = -\left((1-\mu)\frac{x+\mu}{r_1^3}+\mu\frac{x-1+\mu}{r_2^3}\right),$$

$$\ddot{y}+2\dot{x}-y = -y\left((1-\mu)\frac{1}{r_1^3}+\mu\frac{1}{r_2^3}\right)$$

$$\ddot{z} = -z\left((1-\mu)\frac{1}{r_1^3}+\mu\frac{1}{r_2^3}\right)$$

Autonomous
3 DoF
+ Energy integral

The Circular Restricted Three Body Problem (CRTBP)



Inertial OXYZ $\rightarrow$ Rotating Oxyz $\left(\dot{\theta}=\omega=1\right)$

$$X = x\cos t - y\sin t \qquad \dot{X} = \left(\dot{x}-y\right)\cdot\cos t - \left(\dot{y}+x\right)\cdot\sin t$$

$$Y = x\sin t + y\cos t \qquad \dot{Y} = \left(\dot{y}+x\right)\cdot\cos t + \left(\dot{x}-y\right)\cdot\sin t$$

$$Z = z \qquad \dot{Z} = \dot{z}$$

$$x_1 = -\mu, \quad y_1 = z_1 = 0$$

$$x_2 = 1-\mu, \quad y_2 = z_2 = 0$$

$$L = T - V = \frac{1}{2}(\dot{X}^2+\dot{Y}^2+\dot{Z}^2)+V$$

$$= \frac{1}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+(x\dot{y}-\dot{x}y)+\frac{1}{2}\left(x^2+y^2\right)+\frac{1-\mu}{r_1}+\frac{\mu}{r_2}$$

$$r_1 = \sqrt{(x+\mu)^2+y^2+z^2},$$

$$r_2 = \sqrt{(x-1+\mu)^2+y^2+z^2}$$

$$\ddot{x}-2\dot{y} = \frac{\partial U}{\partial x},$$

$$\ddot{y}+2\dot{x} = \frac{\partial U}{\partial y}$$

$$\ddot{z} = \frac{\partial U}{\partial y}$$

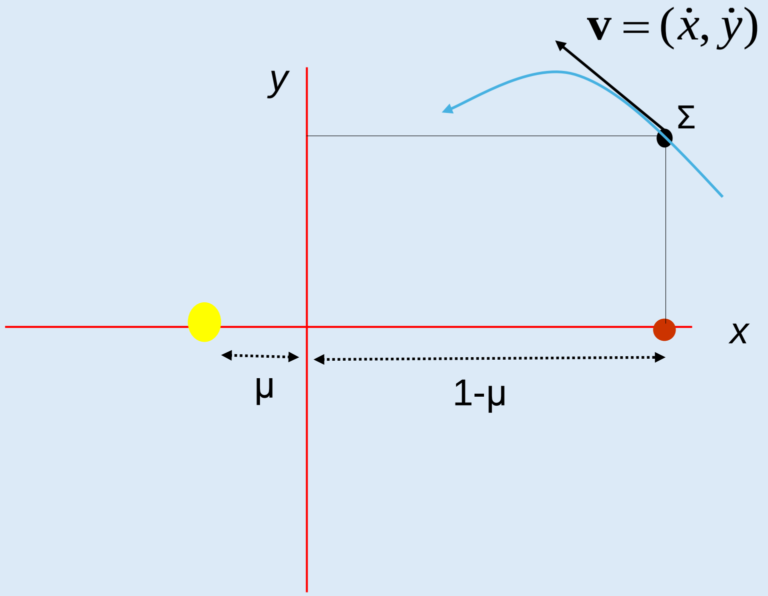
$$U = \frac{\left(x^2+y^2\right)}{2}+\frac{1-\mu}{r_1}+\frac{\mu}{r_2}$$

Potential function

$$C_J = \left(x^2+y^2\right)+2\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)-\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) = -h/2$$

The Jacobi integral

The Planar Circular Restricted Three Body Problem (PCRTBP)



$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x},$$
$$\ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y}$$

$(z = \dot{z} = 0)$

Autonomous
2 DoF

$$h = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \left(\frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right) - \frac{1}{2}\left(x^2 + y^2\right)$$

- Equilibrium points
- Bounds of motion
- Poincare Sections
- Periodic Orbits

PCRTBP – Lagrange points

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x} = 0 \qquad x - \left((1-\mu)\frac{x+\mu}{r_1^3} + \mu \frac{x-1+\mu}{r_2^3} \right) = 0, \quad (1)$$

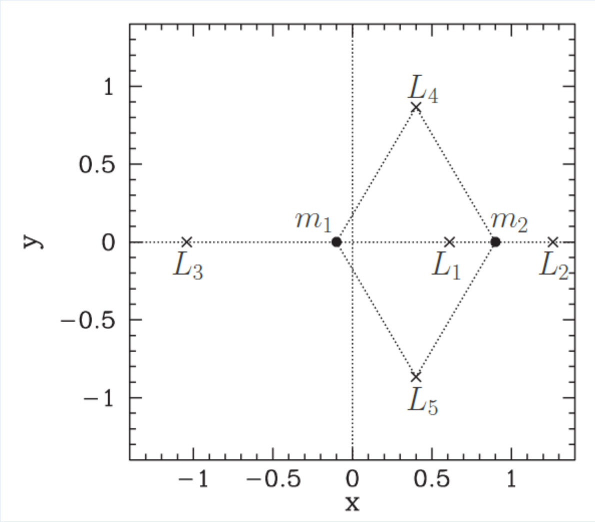
$$\ddot{y} + 2\cdot\dot{x} = \frac{\partial U}{\partial y} = 0 \qquad y\left[1 - (1-\mu)\frac{1}{r_1^3} - \mu\frac{1}{r_2^3} \right] = 0 \quad (2)$$

$$\Downarrow$$

L1, L2, L3 : $y=0$, $x=x(\mu)$ root of

$$x - (1-\mu)\frac{x+\mu}{|x+\mu|^3} - \mu\frac{x-1+\mu}{|x-1+\mu|^3} = 0$$

$r_1 = r_2 = 1 \rightarrow$ L4, L5 $\quad x = \frac{1}{2} - \mu, \quad y = \pm\frac{\sqrt{3}}{2}$

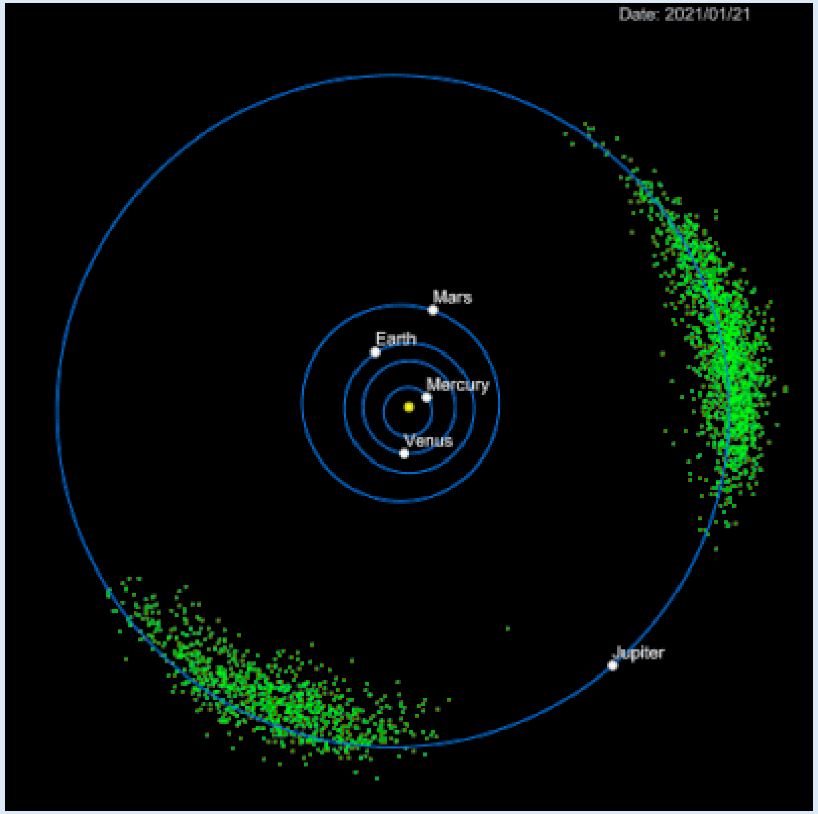


Stability

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ U_{xx} & U_{xy} & 0 & 2 \\ U_{xy} & U_{yy} & -2 & 0 \end{pmatrix}_{L_i}$$

Jacobian matrix of the planar system

C.E. $\gamma^4 + (4 + U_{xx} + U_{yy})\gamma^2 + (U_{xx}U_{yy} - U_{xy}^2) = 0.$



L_1, L_2, L_3 : Unstable

L_4, L_5 : Stable for

$$\mu < \frac{1}{2} - \frac{\sqrt{69}}{18} \approx 0.0385$$

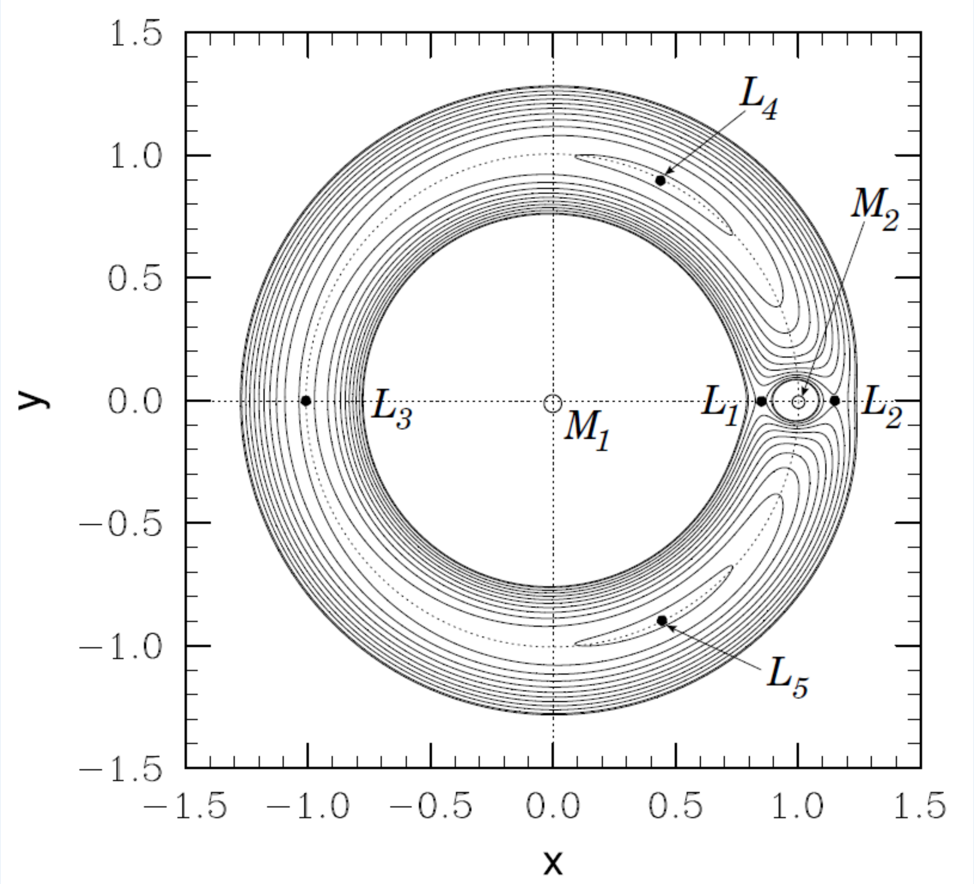
(The Trojan asteroids)

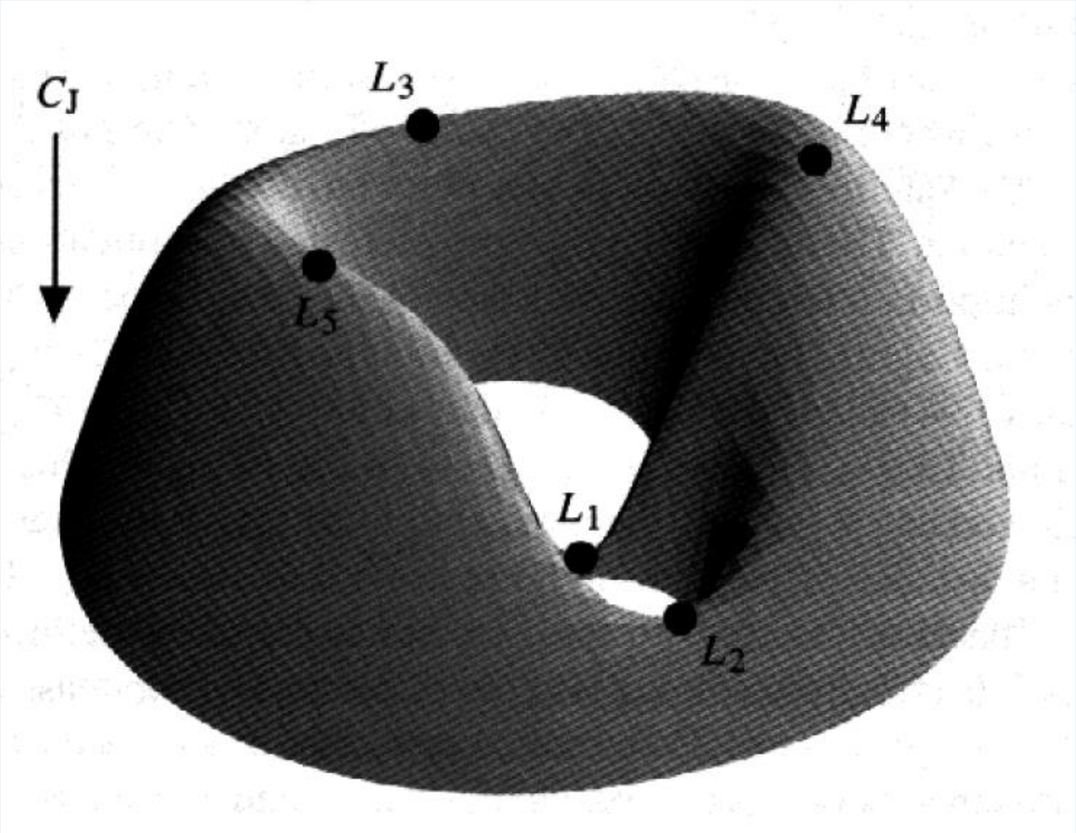
PCRTBP – Bounds of motion

$$\frac{1}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)=h+\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)+\frac{1}{2}\left(x^2+y^2\right)\geq 0$$

$$h+\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)+\frac{1}{2}\left(x^2+y^2\right)=0$$

Zero velocity curves

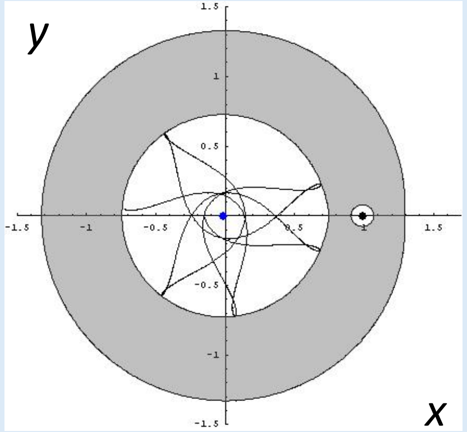




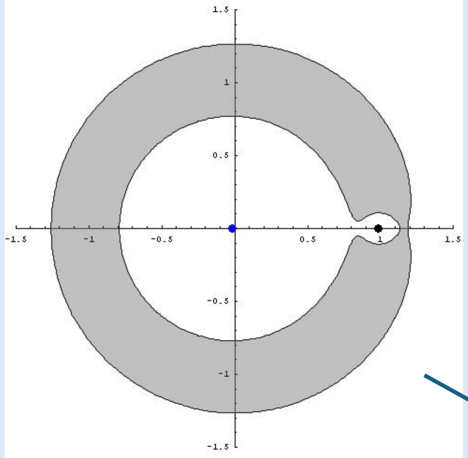
PCRTBP – Bounds of motion

μ=0.0123 (Earth-Moon)

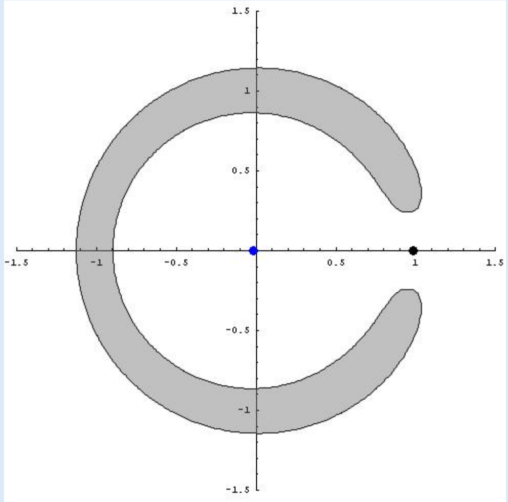
$h < -1.594775$



$-1.594775 < h < -1.586575$

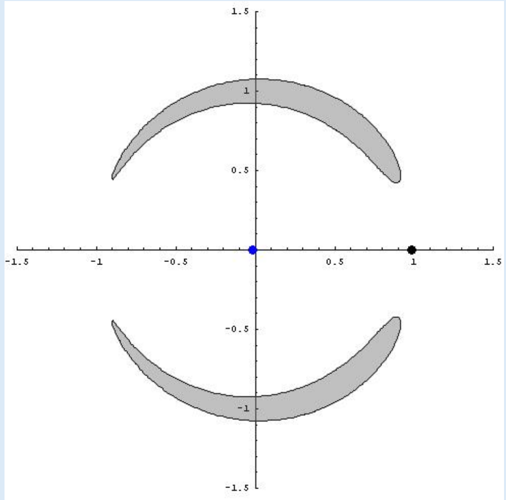


$-1.586575 < h < -1.506395$

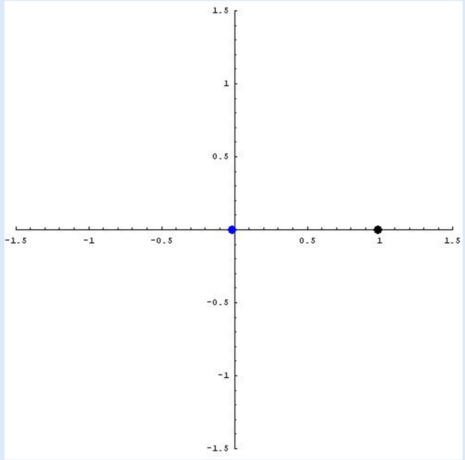


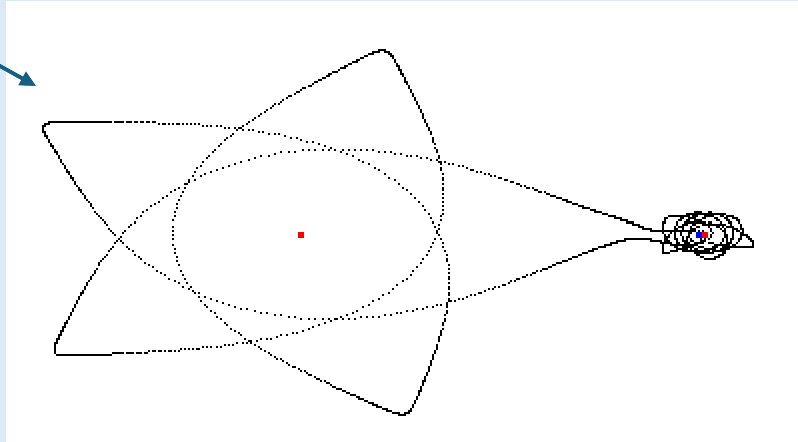
Earth-Moon System
μ=0.0123

$-1.506395 < h < -1.49385$



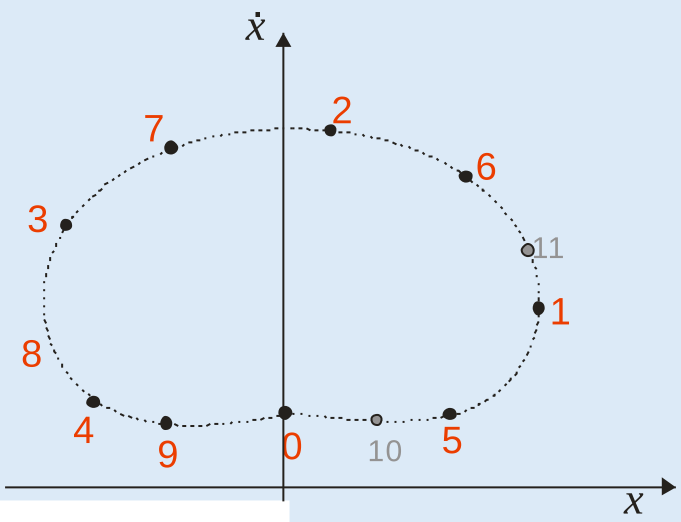
$h > -1.49385$

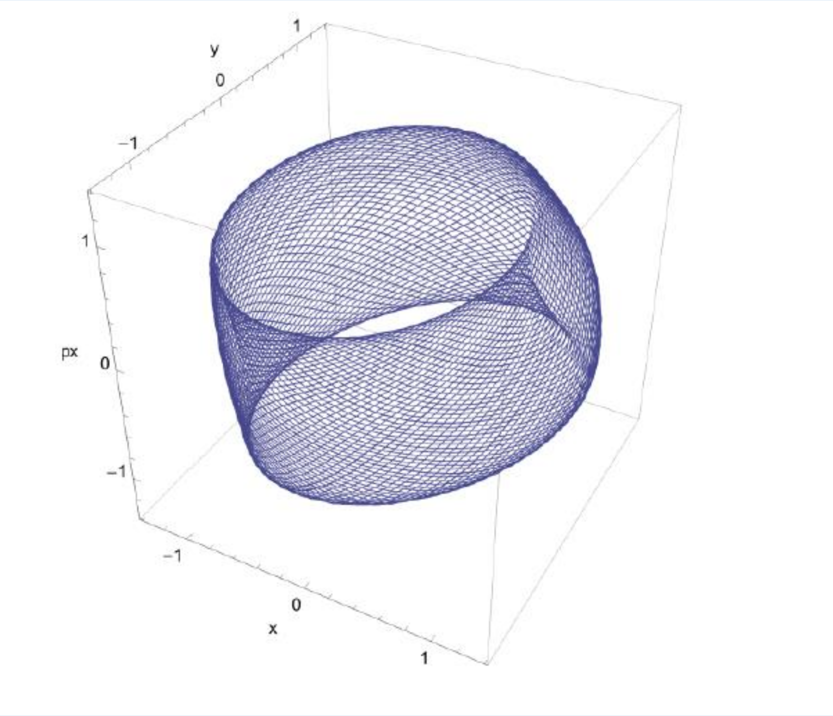


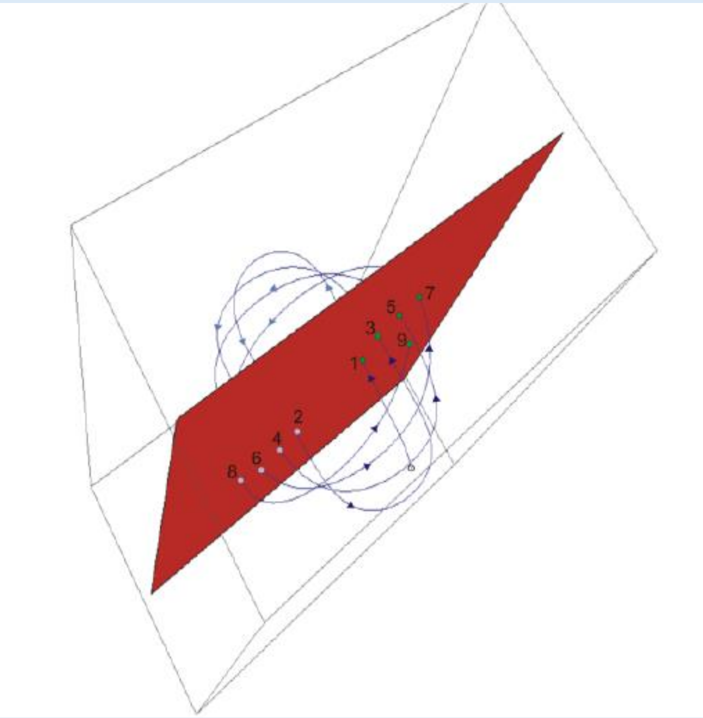


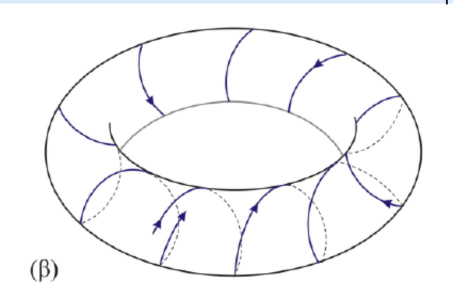
PCRTBP – Poincare Sections

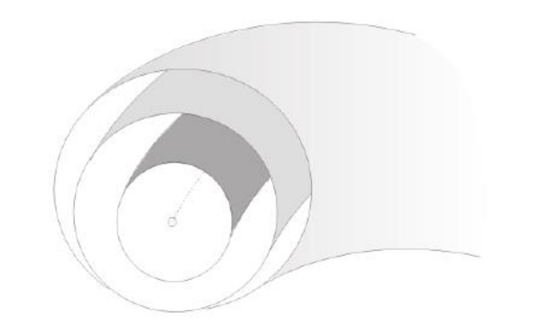
$$\begin{aligned} x &= x(t) \\ y &= y(t) \\ \dot{x} &= \dot{x}(t) \\ \dot{y} &= \dot{y}(t) \end{aligned} \quad \xrightarrow{h=const} \quad \begin{aligned} x &= x(t) \\ y &= y(t) \\ \dot{x} &= \dot{x}(t) \\ \dot{y} &= f(x,y,\dot{x},h) \end{aligned} \quad \xrightarrow{\text{section } y=0} \quad \begin{aligned} x &= x(t) \\ y &= 0 \\ \dot{x} &= \dot{x}(t) \\ \dot{y} &= f(x,y,\dot{x},h) > 0 \end{aligned}$$







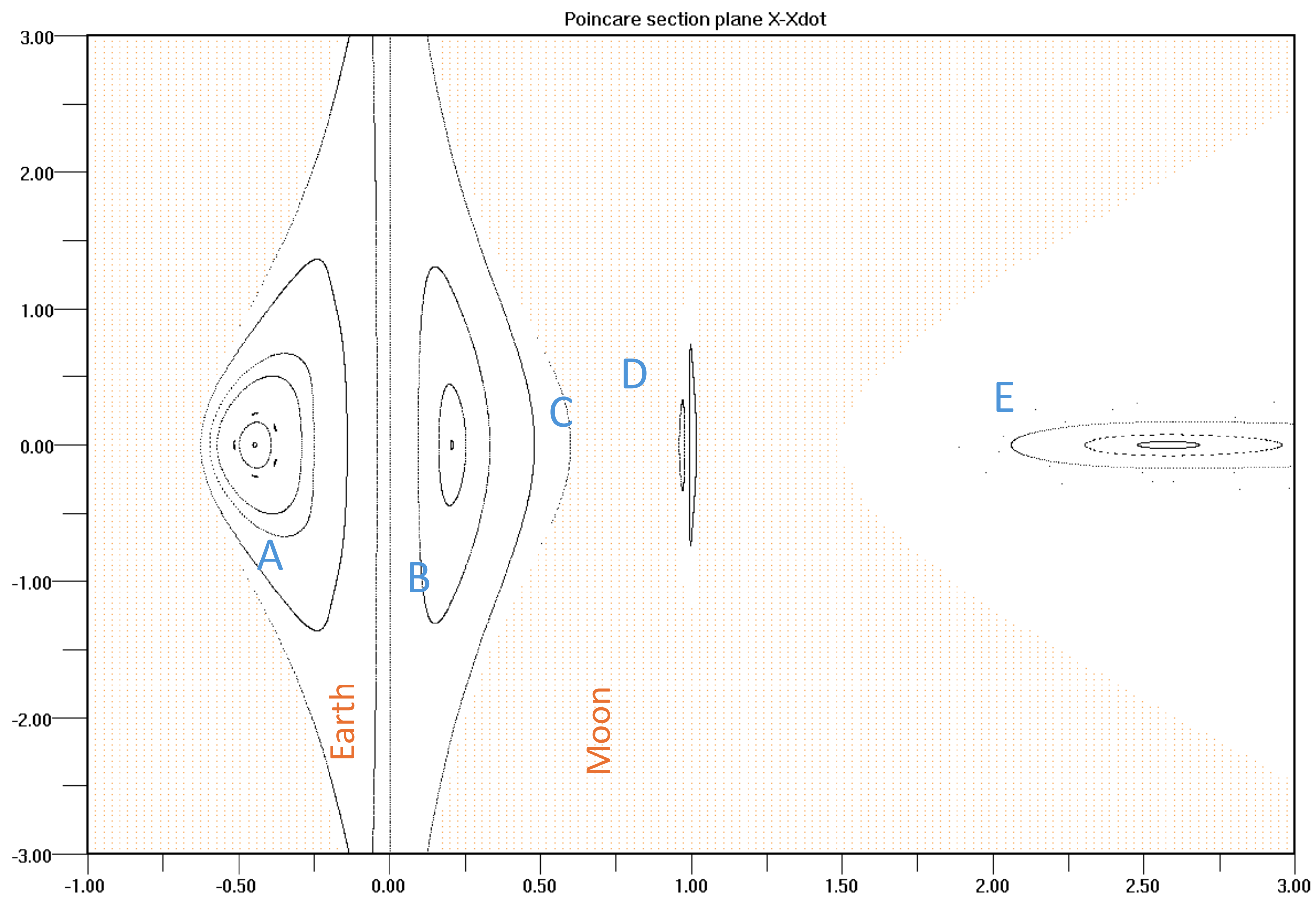




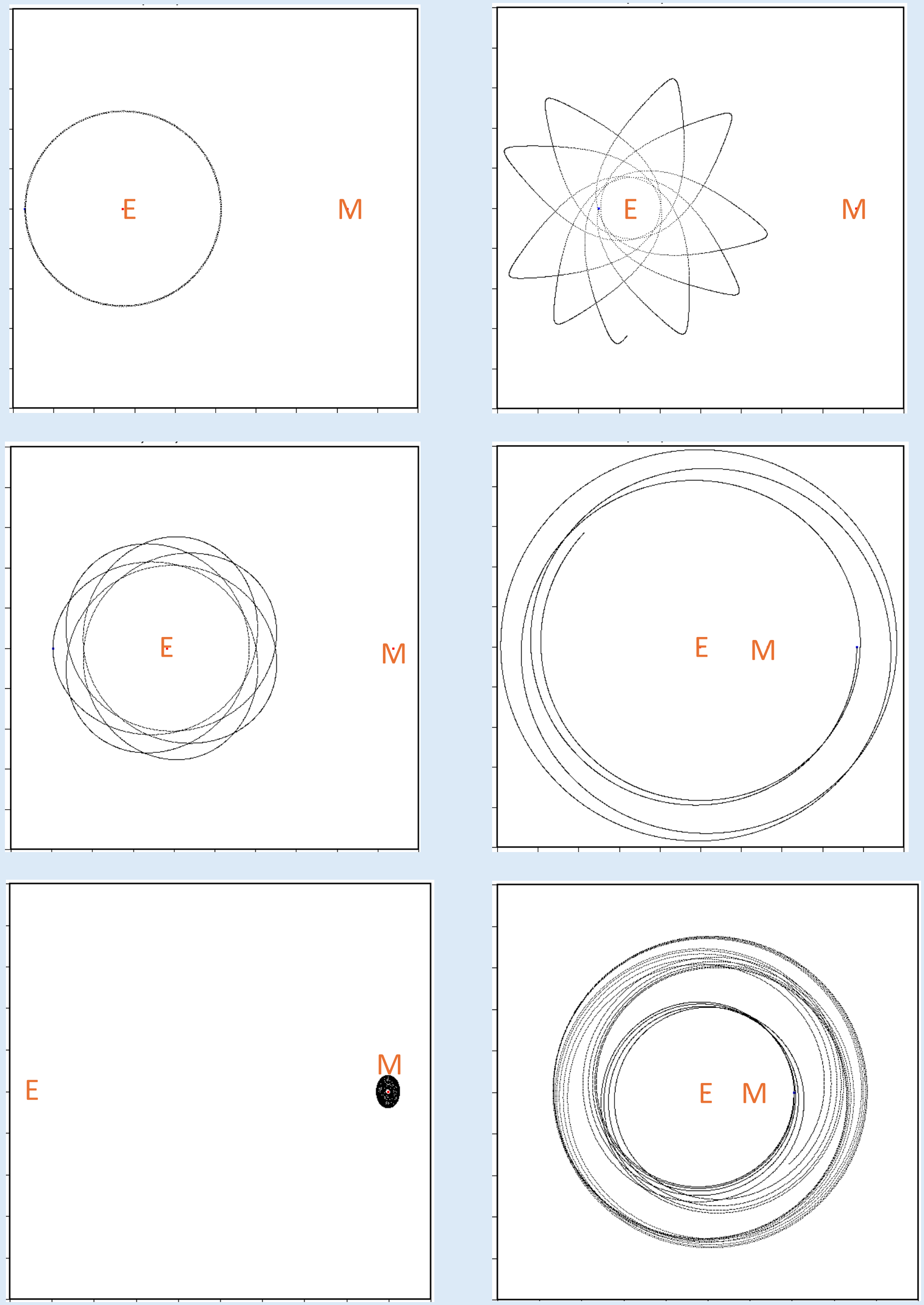
- Quasi-periodic orbits
- Periodic Orbits
- Chaotic Orbits

PCRTBP – Poincare Sections

μ=0.0123, ***h*=-1.80** $y=0, \quad \dot{y}<0$

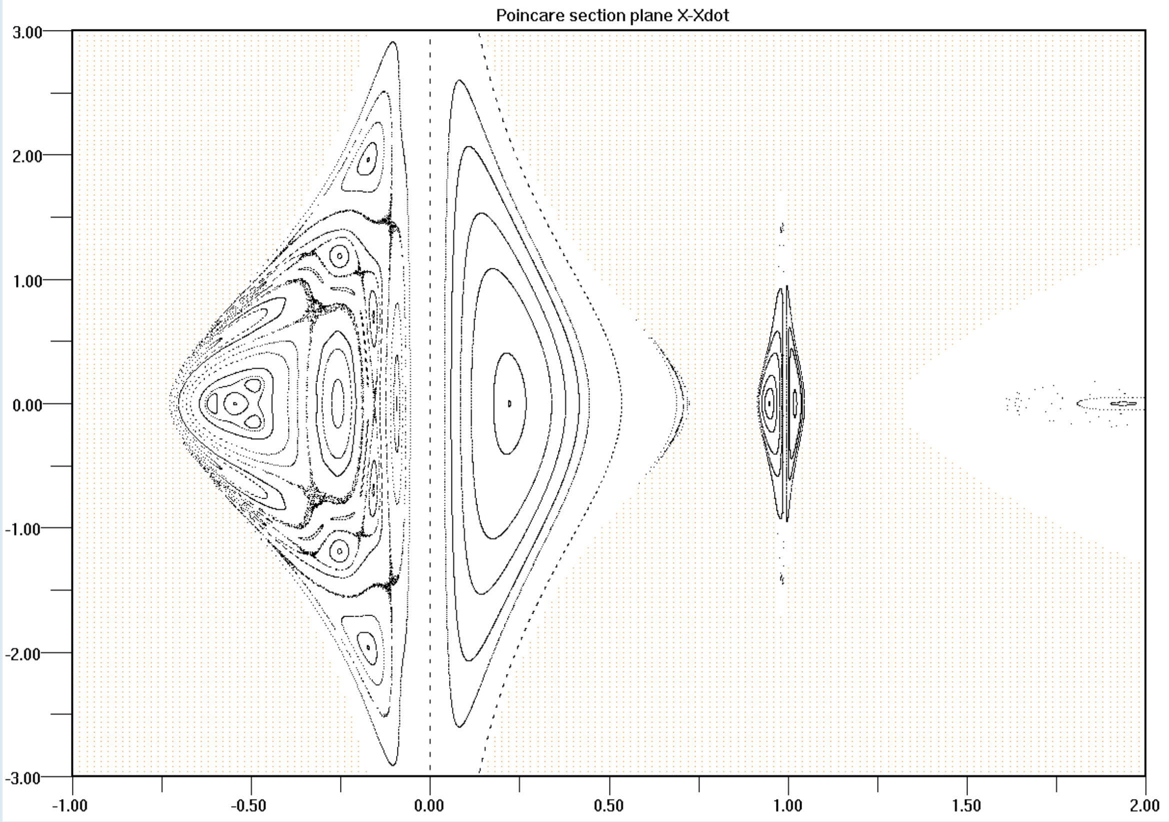


1. Prograde orbits around main primary
2. Retrograde orbits around main primary
3. Prograde orbits around secondary
4. Retrograde orbits around secondary
5. Outer orbits (prograde or retrograde)



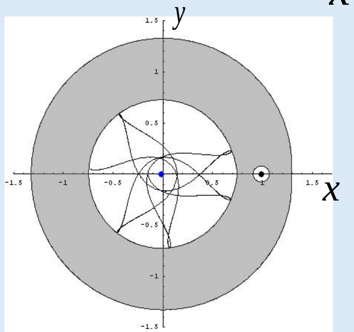
PCRTBP – Poincare Sections

μ=0.0123, ***h*=-1.65**

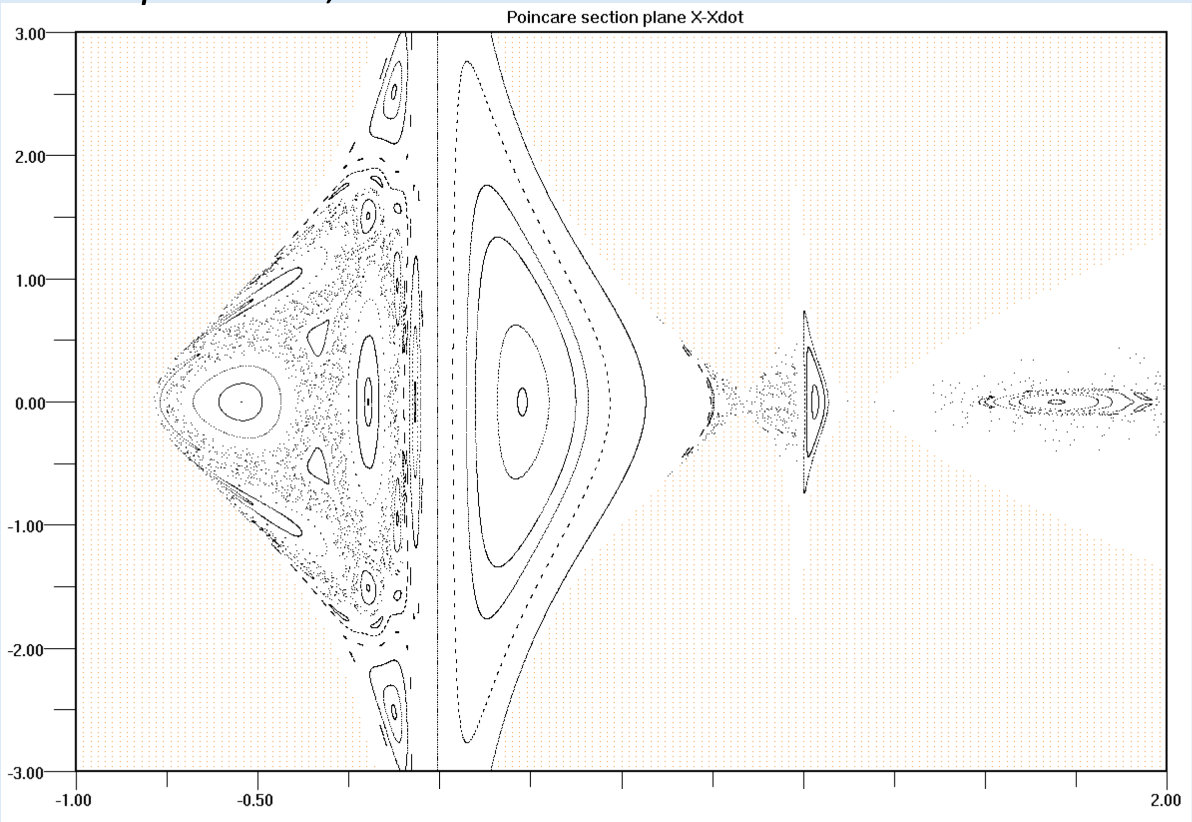


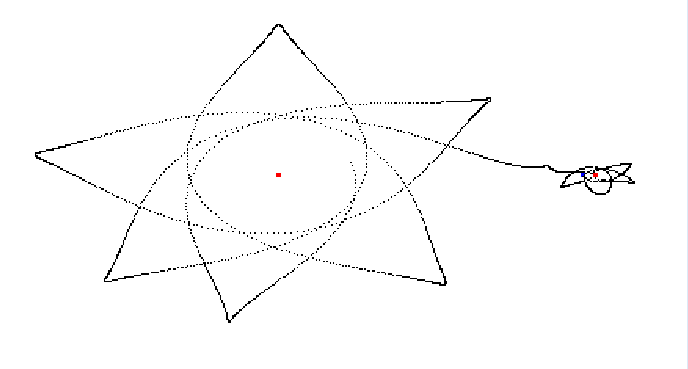
$\dot{x}$

x



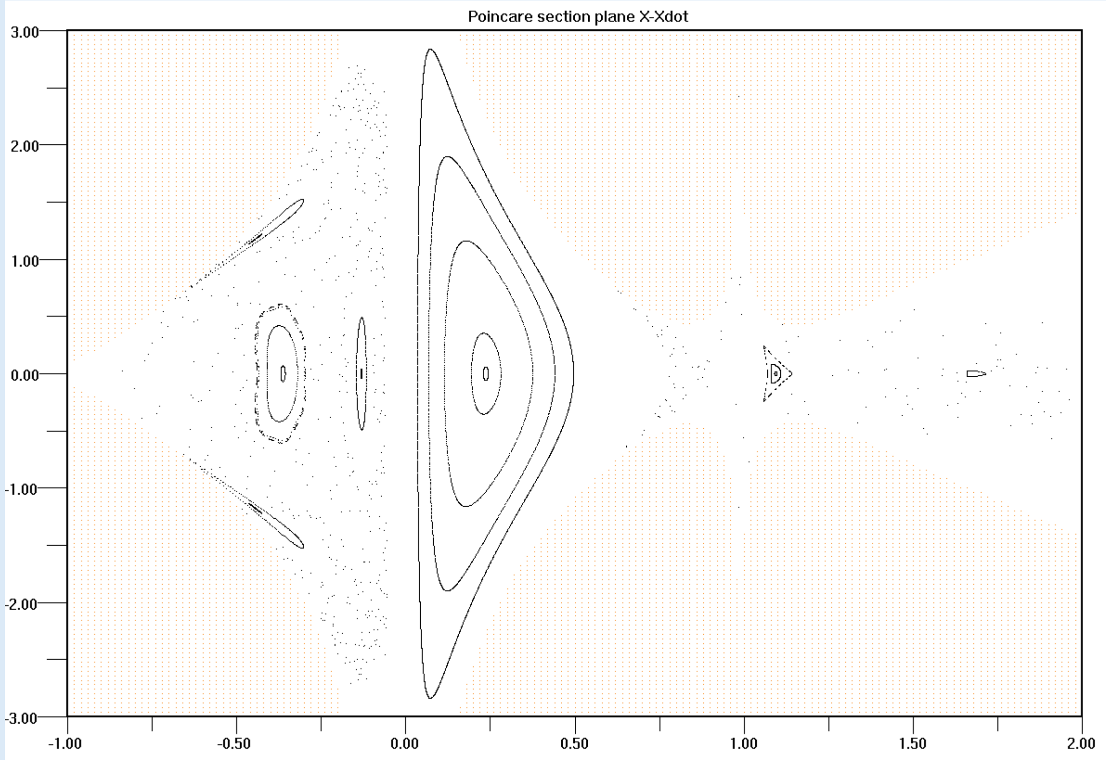
μ=0.0123, ***h*=-1.59**



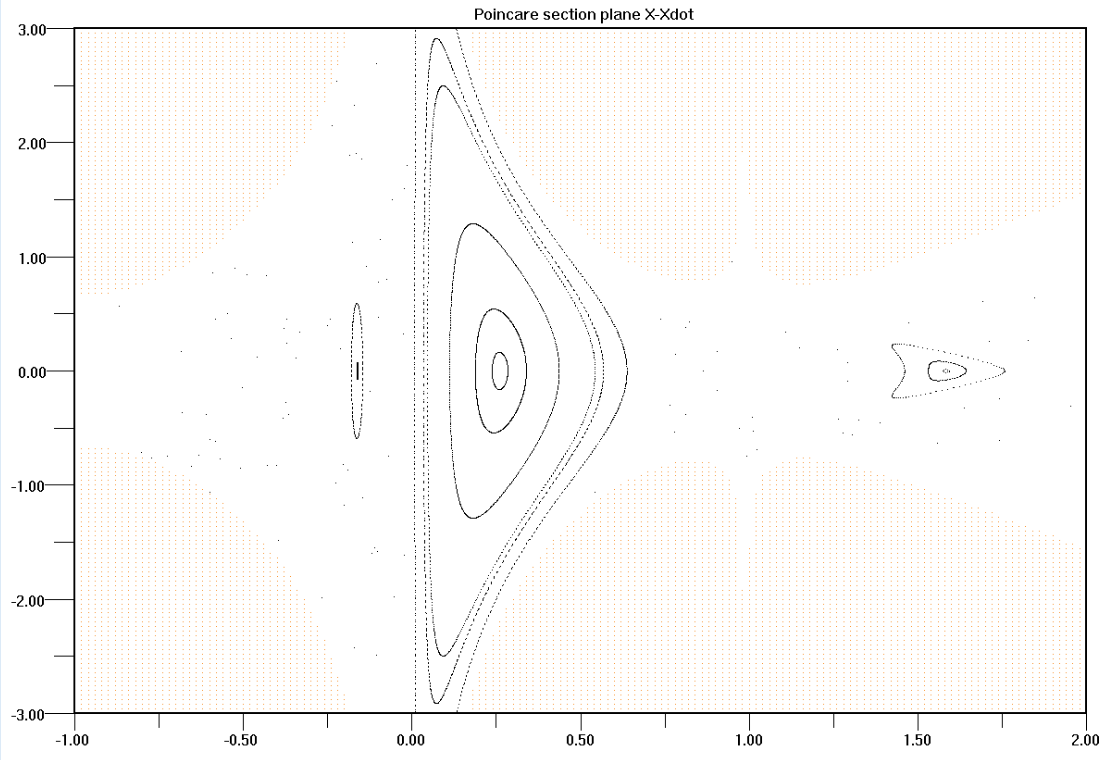


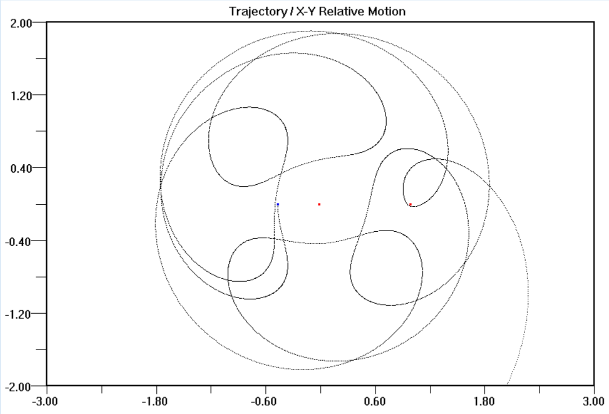
PCRTBP – Poincare Sections

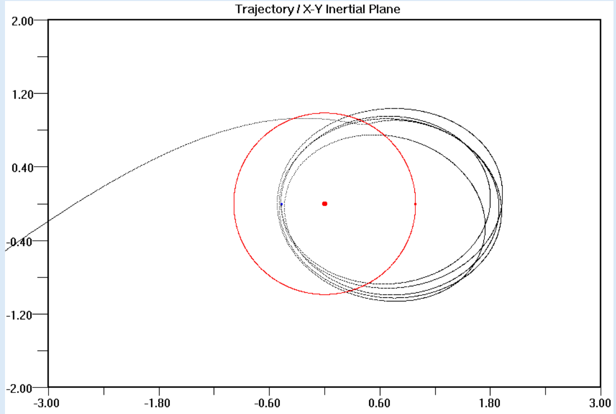
μ=0.0123, ***h*=-1. 5**



μ=0.0123, ***h*=-1. 3**

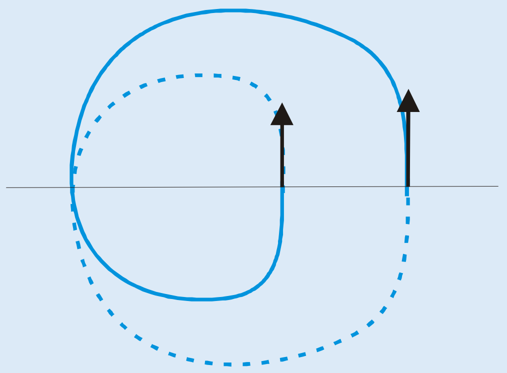




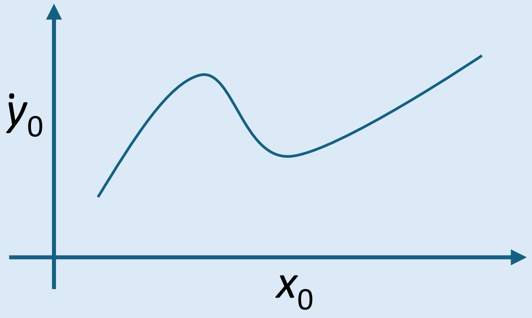


PCRTBP – Symmetric Periodic Orbits

- EoM are invariant under the symmetry $\Sigma = \{x \to x, \quad y \to -y, \quad t \to -t\}$
  - The orbits with initial conditions A) $x_0, y_0, \dot{x}_0, \dot{y}_0$ B) $x_0, -y_0, -\dot{x}_0, \dot{y}_0$ are symmetric with respect to the *x-axis*
  - If an orbit has two perpendicular crossings with the *x-axis* then it is a **symmetric periodic orbit**



- Initial conditions of a symmetric periodic orbit : $\{x_0, \, y_0 = 0, \, \dot{x}_0 = 0, \, \dot{y}_0\}$
- Multiplicity (*m*): number of sections with x-axis (to the same direction)
- Periodicity Condition $\dot{x}_0(t^*) = 0$ here *t** the time of the *m-section.*
- Period *T*=2*t**



Generally, periodic orbits are continued with parameter x_0 (or *h*) forming characteristic curves (**families**) in the plane $(x_0, \dot{y}_0)$

PCRTBP – Stability of Periodic Orbits

$\mathbf{X} = (x_1, x_2, x_3, x_4) = (x, y, \dot{x}, \dot{y})$

$\mathbf{X}_0 = (x_{10}, x_{20}, x_{30}, x_{40})$ p.o. initial condition

$\mathbf{Y} = (y_1, y_2, y_3, y_4)$ variations

$$\dot{\mathbf{Y}} = \mathbf{A} \cdot \mathbf{Y} \quad \Rightarrow \quad \mathbf{Y}(t) = \mathbf{\Delta}(t)\mathbf{Y}_0$$

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ U_{xx} & U_{xy} & 0 & 2 \\ U_{xy} & U_{yy} & -2 & 0 \end{pmatrix}_{L_i}$$

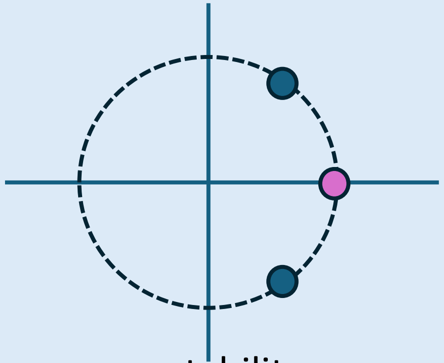
$$\mathbf{\Delta}(t) = \left\{ \bar{y}_{ij} = \frac{\partial x_i(t)}{\partial x_{i0}} \right\}, \qquad \mathbf{\Delta}(0) = \mathbf{I}$$

(*Matrizant* or *State Transition Matrix*)

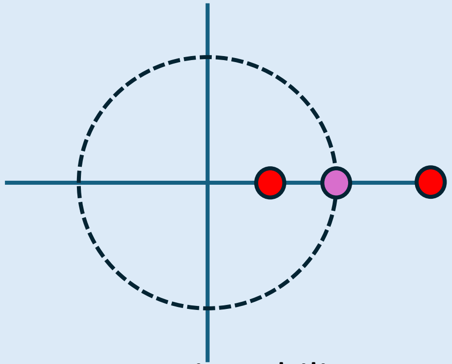
Monodromy matrix : $\mathbf{\Delta}(T) \rightarrow \quad \sigma_i = \dfrac{\ln \lambda_i}{T}$ (C.E.)

* $\mathbf{\Delta}(T)$ is symplectic with eigenvalues

$$\lambda_1 = \lambda_2 = 1, \qquad \lambda_3 = \lambda_4^* \quad \text{or} \quad \lambda_3 = 1/\lambda_4$$



stability

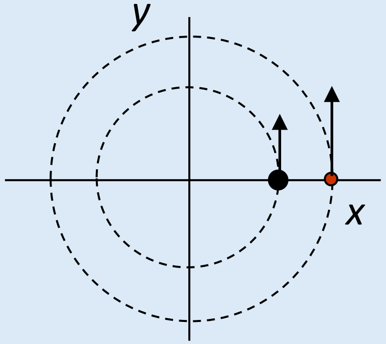


instability

Reduced Monodromy matrix : $\mathbf{M} = \begin{bmatrix} \dfrac{\partial x_1}{\partial x_{10}} & \dfrac{\partial x_1}{\partial x_{40}} \\ \dfrac{\partial x_4}{\partial x_{10}} & \dfrac{\partial x_4}{\partial x_{40}} \end{bmatrix}_{t=T}$

- |trace**M**|<2 : stability
- |trace**M**|<2 : instability

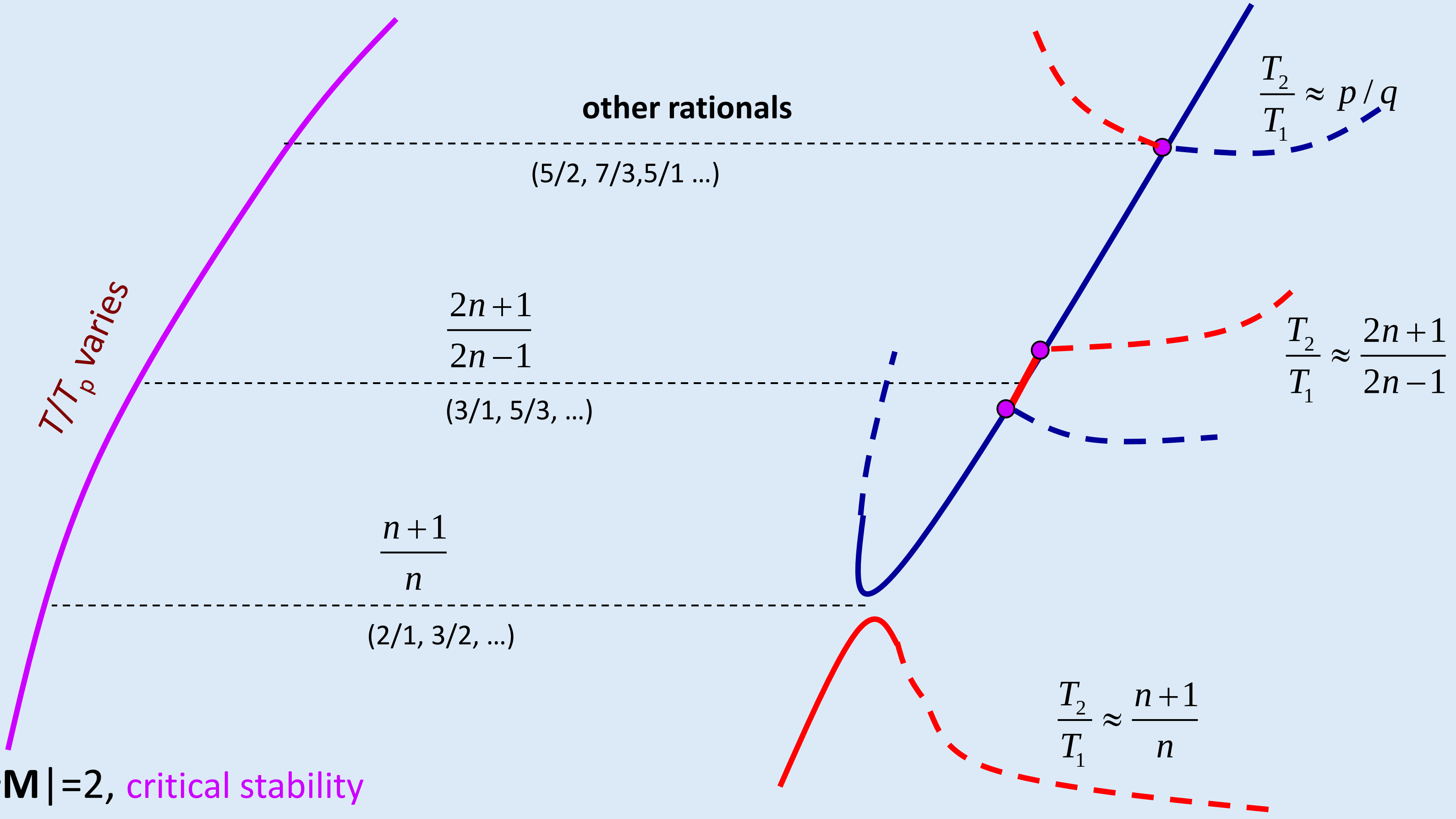
PCRTBP – Families of Periodic Orbits (sketch)



Circular orbits of the unperturbed system (**μ=0**)

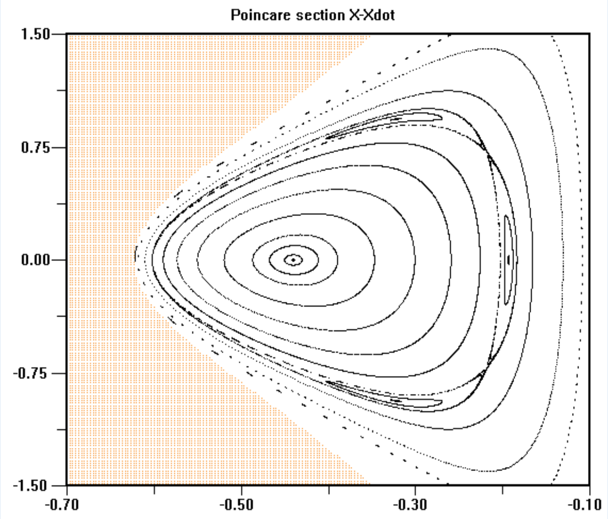
Resonant periodic orbit : $\dfrac{T}{T_p} = \dfrac{p}{q}, \quad p,q \in N$

Almost circular orbits
+ Elliptic orbits of the perturbed system (**$\mu \neq 0$**),

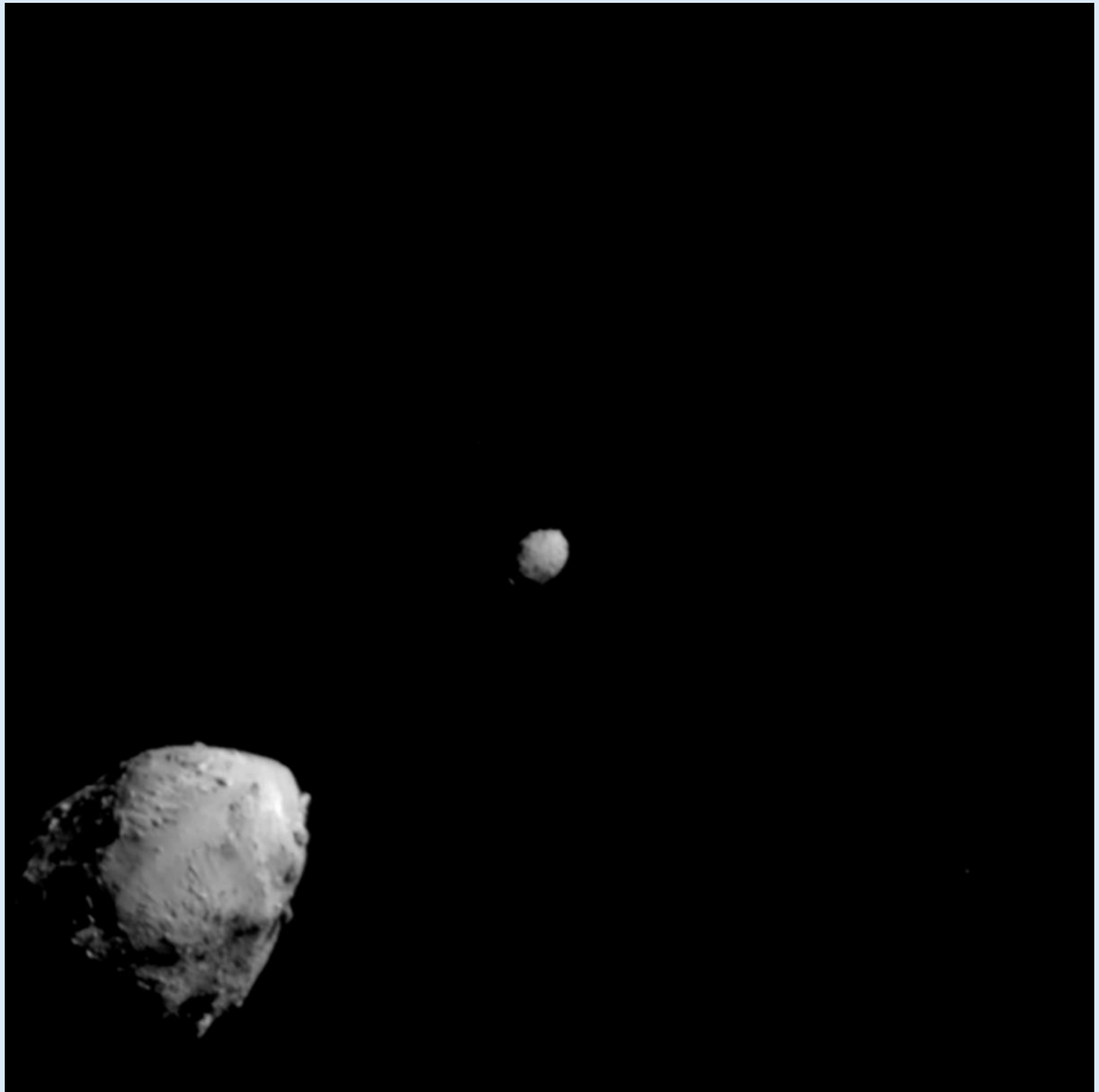


$|trace\mathbf{M}|$=2, critical stability

Elliptic orbits are resonant (Mean Motion Resonances)



Application:
The Didymos-Dimorphos binary asteroid system

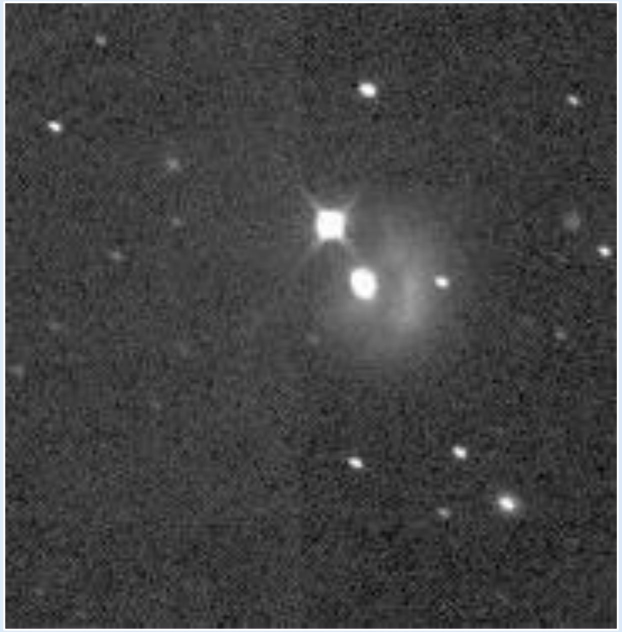


Parameters of the Didymos-Dimorphos system			
physical units			normalized units
μ=m2/(m1+m2)			**0.007515**
d	1152	m	**1**
a1	405	m	0.3516
b1	405	m	0.3516
c1	304	m	0.2635
a2	89.5	m	0.0874
b2	84.5	m	0.0672
c2	57.5	m	0.0555
System Period	11.37	h	**2π**
SRP Acceleration	2.70E-08	m/s^2	**0.001**

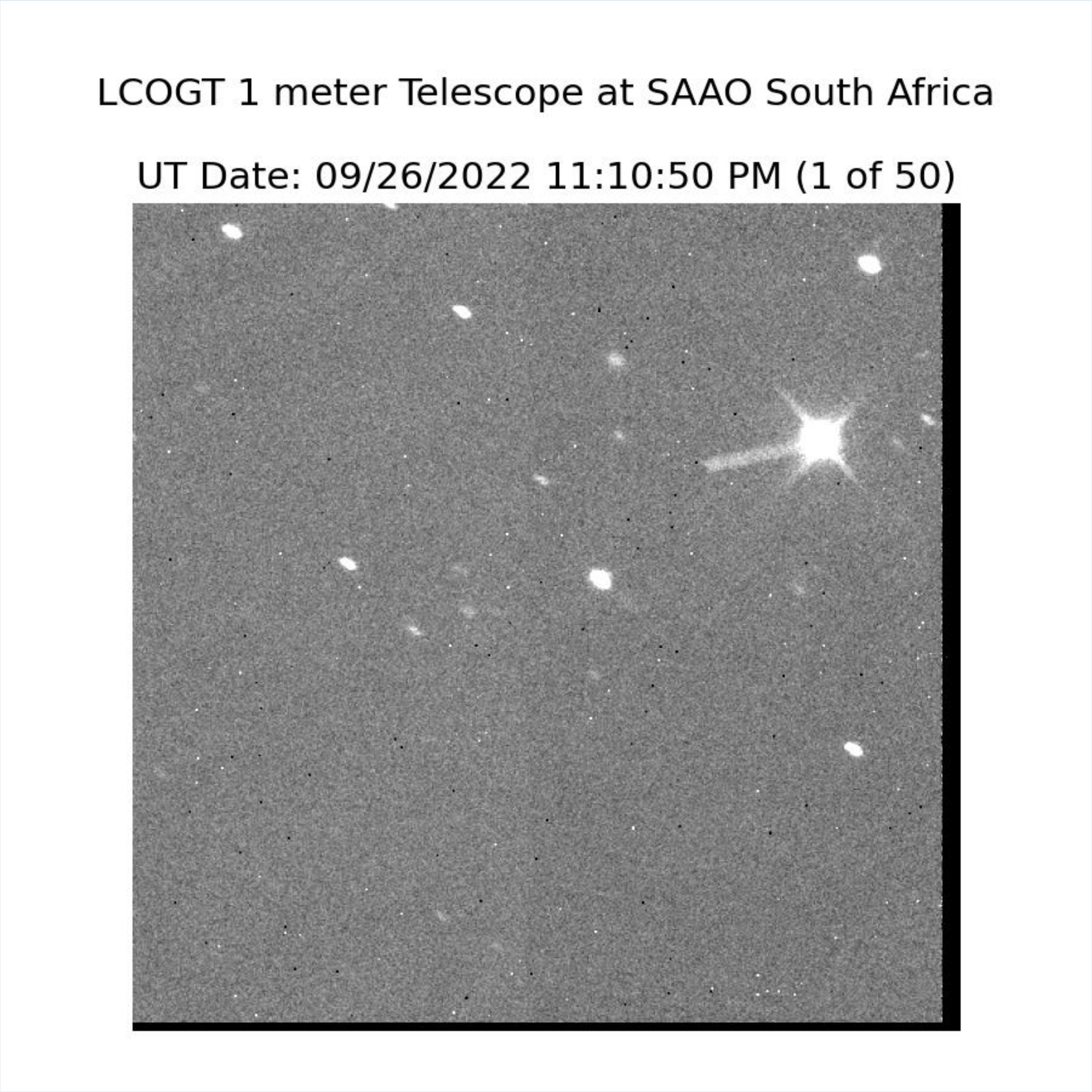
DART mission (NASA – John Hopkins Univ.)
November 2021 – September 2026

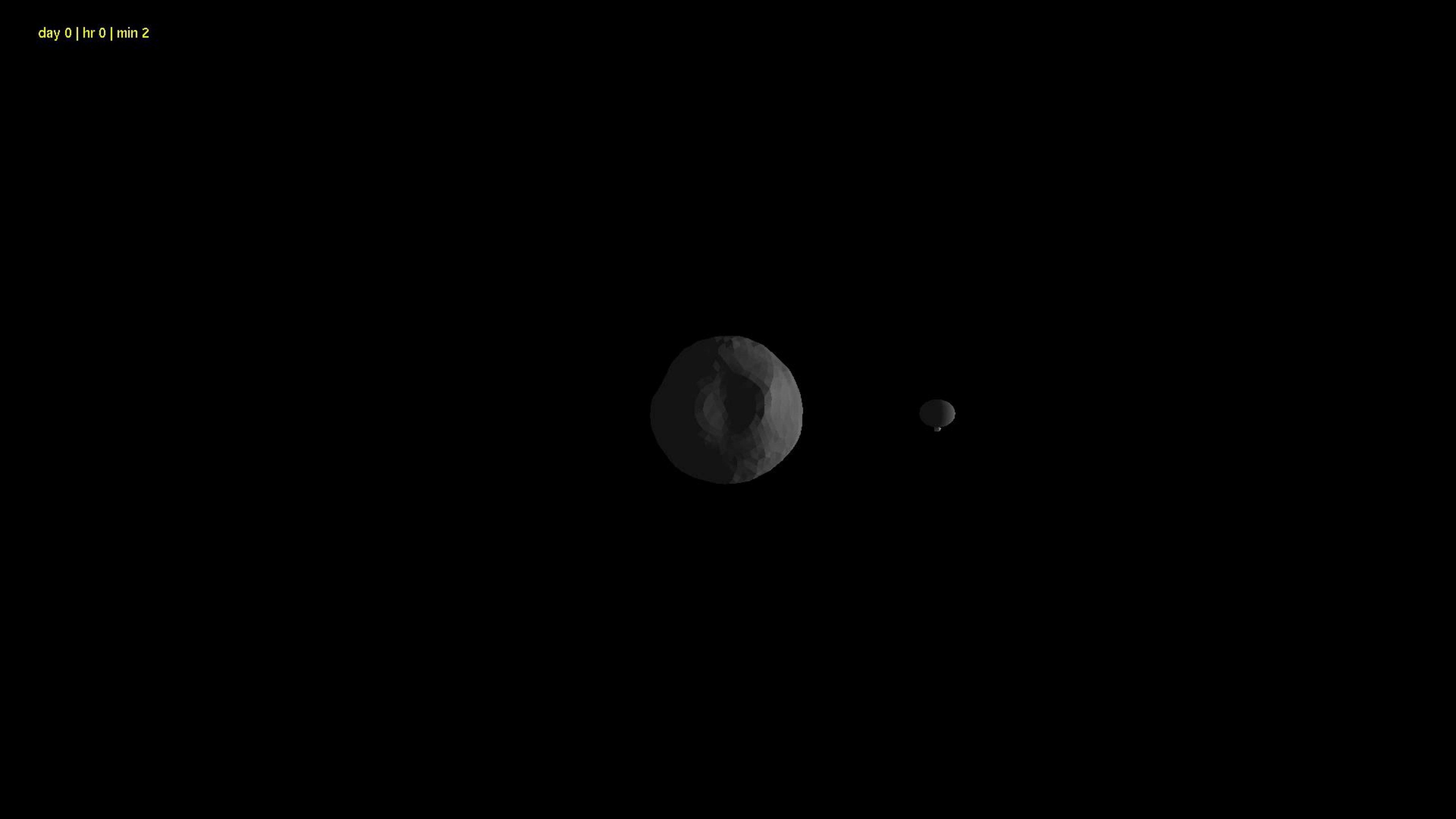
September 26
23:26 UTC
(12 min. post-impact)

23:29 UTC
(15 min. post-impact)

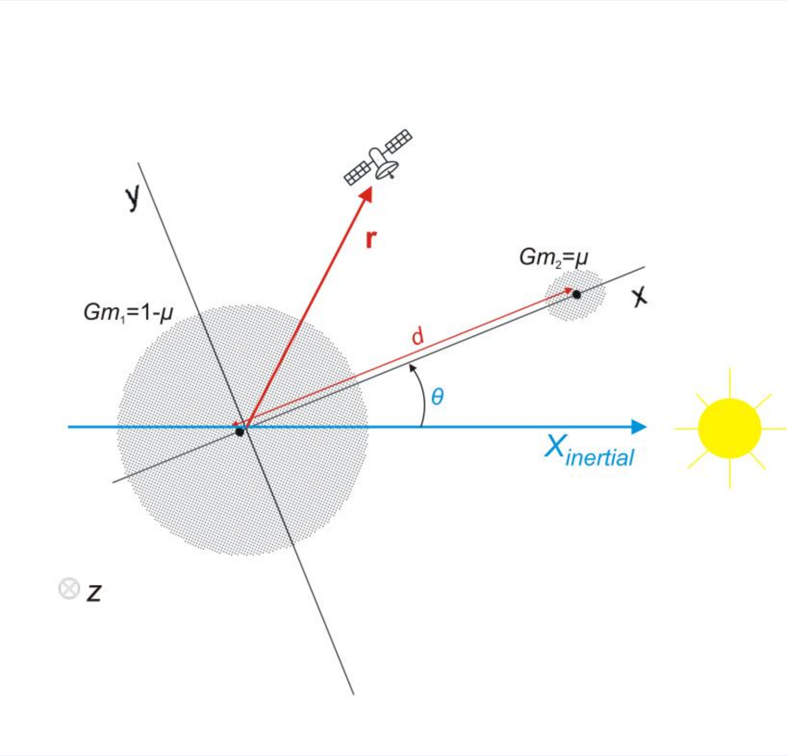


Credit: Tim Lister, Joseph Chatelain, Rachel Street, Edward Gomez, Joseph Farah / Las Cumbres Observatory.




day 0 | hr 0 | min 2

The model



Collisions
(not mathematical singularities)

- Oblate primary (Didymos), ellipsoid secondary (Dimorphos)
- Primaries move in circular motion with constant angular frequency ***ω***
- Rotating frame Oxyz (similar to RTBP)
- Secondary is phase-locked along Ox axis
- Gravitational potentials : Homogeneous triaxial ellipsoids up to 2nd order

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x},$$

$$\ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y}$$

$$U = (1-\mu)U_e(x+\mu, y, z; I_{1x}, I_{1x}, I_{1z}) + \mu U_e(x-1+\mu, y, z; I_{2x}, I_{2x}, I_{2z})$$

$$U_e(x, y, z; I_x, I_y, I_z) = -\frac{1}{r} - \frac{I_x + I_y + I_z}{2r^3} + \frac{3}{2}\frac{I_x x^2 + I_y y^2 + I_z z^2}{r^5}$$

I_{xi}, I_{yi}, I_{zi} : moments of inertia of body *i* ($I_{x1}=I_{y1}$)

Classification of Orbits

(inner orbits)

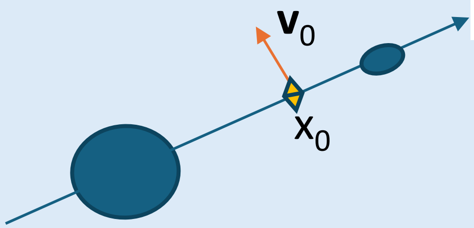
R1 : Retrogrades around primary
D1 : Direct around primary
R2 : Retrogrades around secondary
D2 : Direct around secondary

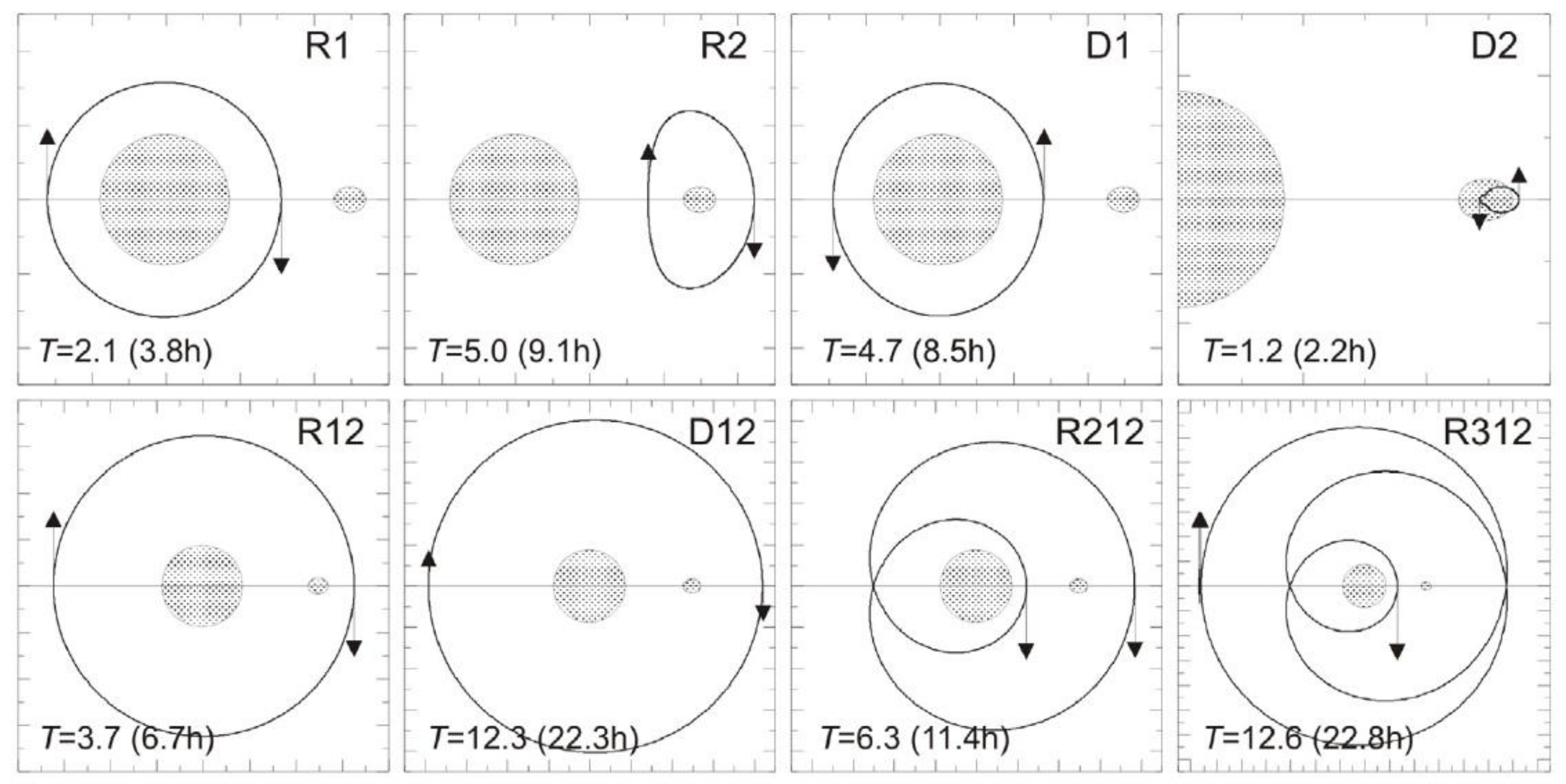
(outer orbits)

R12 : Outer Retrogrades
D12 : Outer Direct

Higher multiplicity orbits = R312,R212

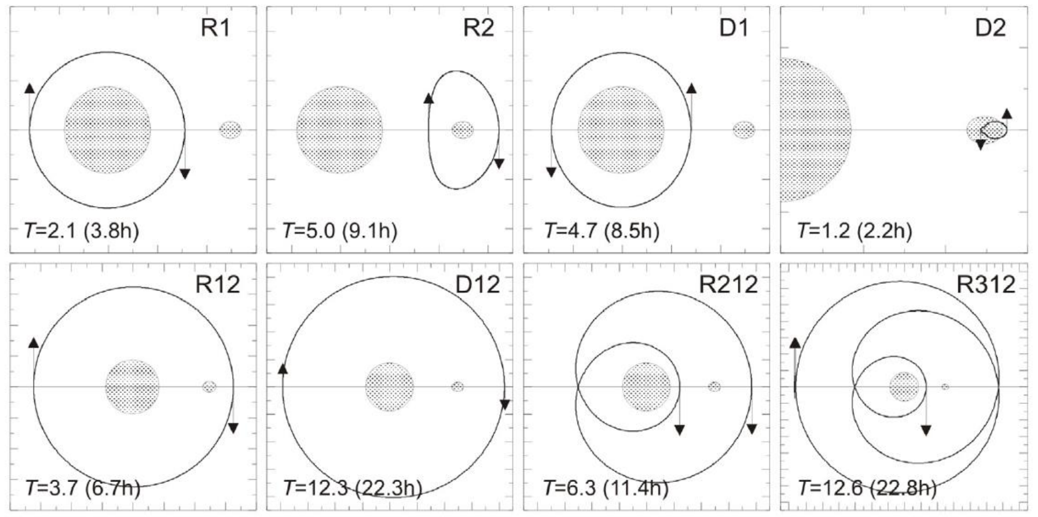
- For all these types of orbits we can assign initial conditions (x_0, y_0=0, v_{x0}=0, v_{y0}) in the rotating frame

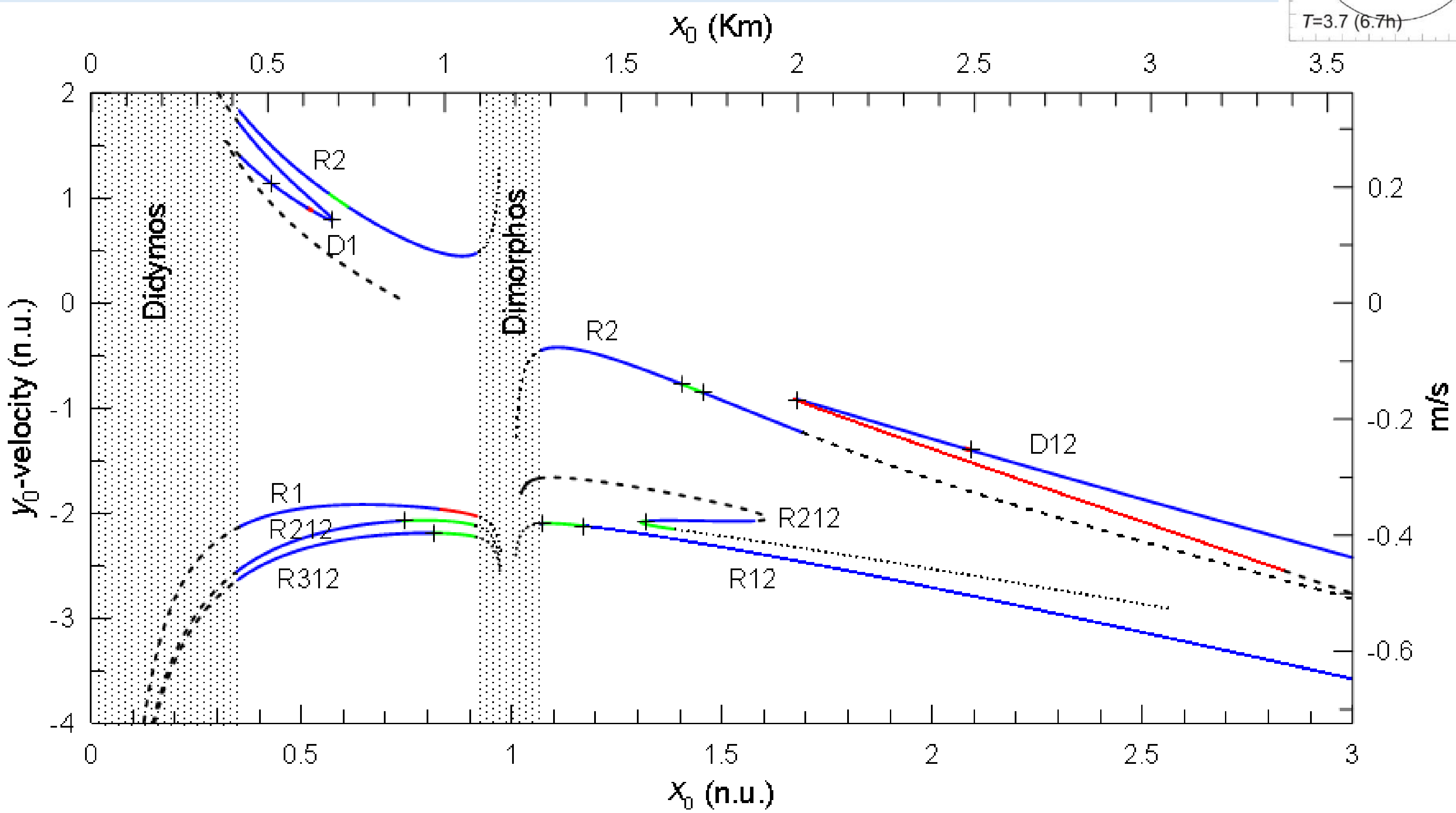




Symmetric periodic orbits

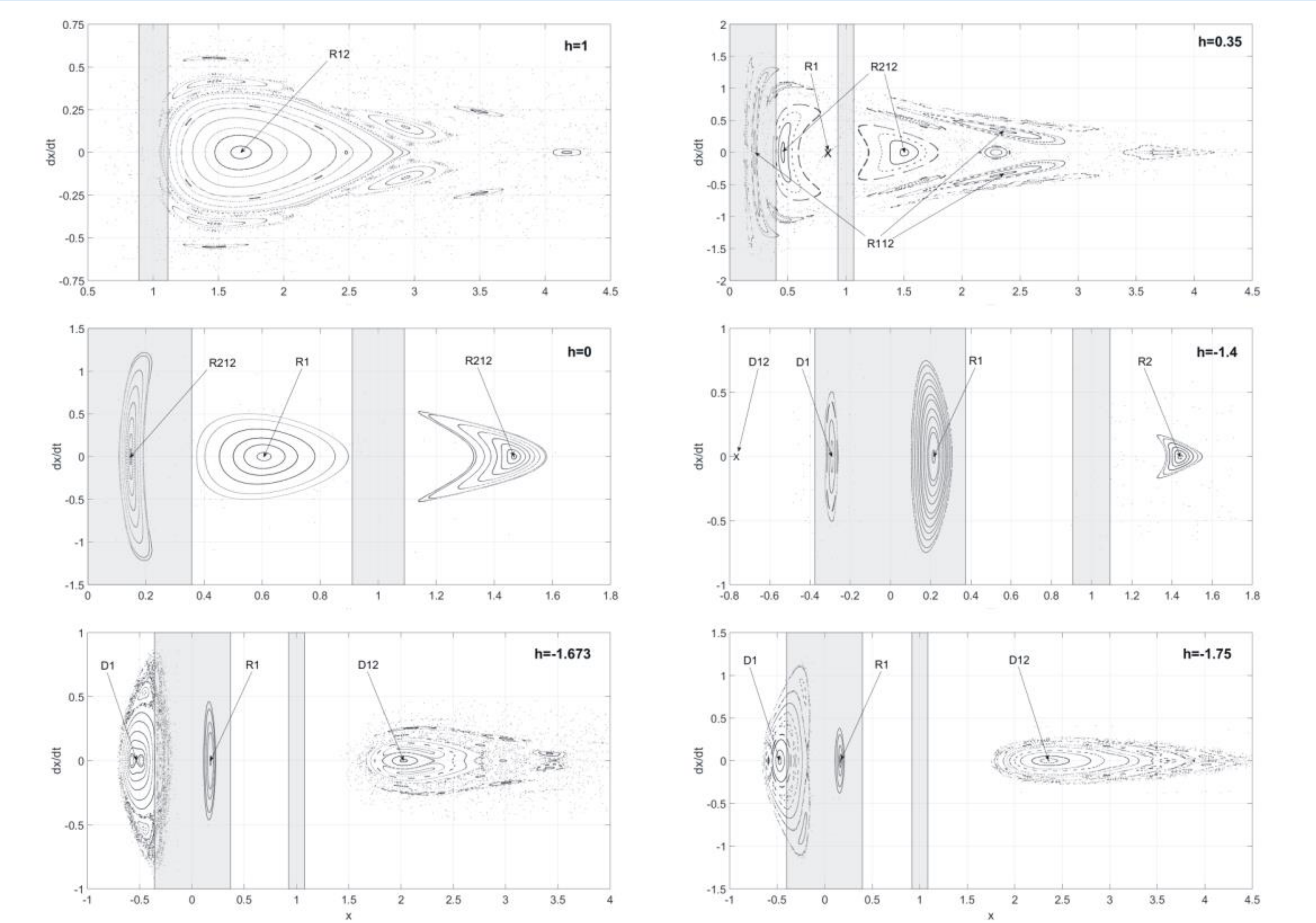
Families of periodic orbits





Voyatzis et al, 2023
Dell'Elce et al, 2017

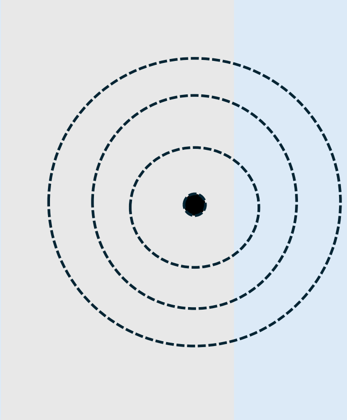
Poincare sections



$(x, \dot{x})$ *plane*

$y = 0, \dot{y} > 0$

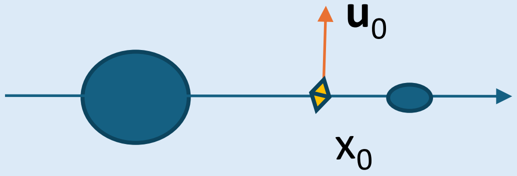
* If a stable periodic orbit is a collision orbit, then all invariant tori around it consist of collision orbits

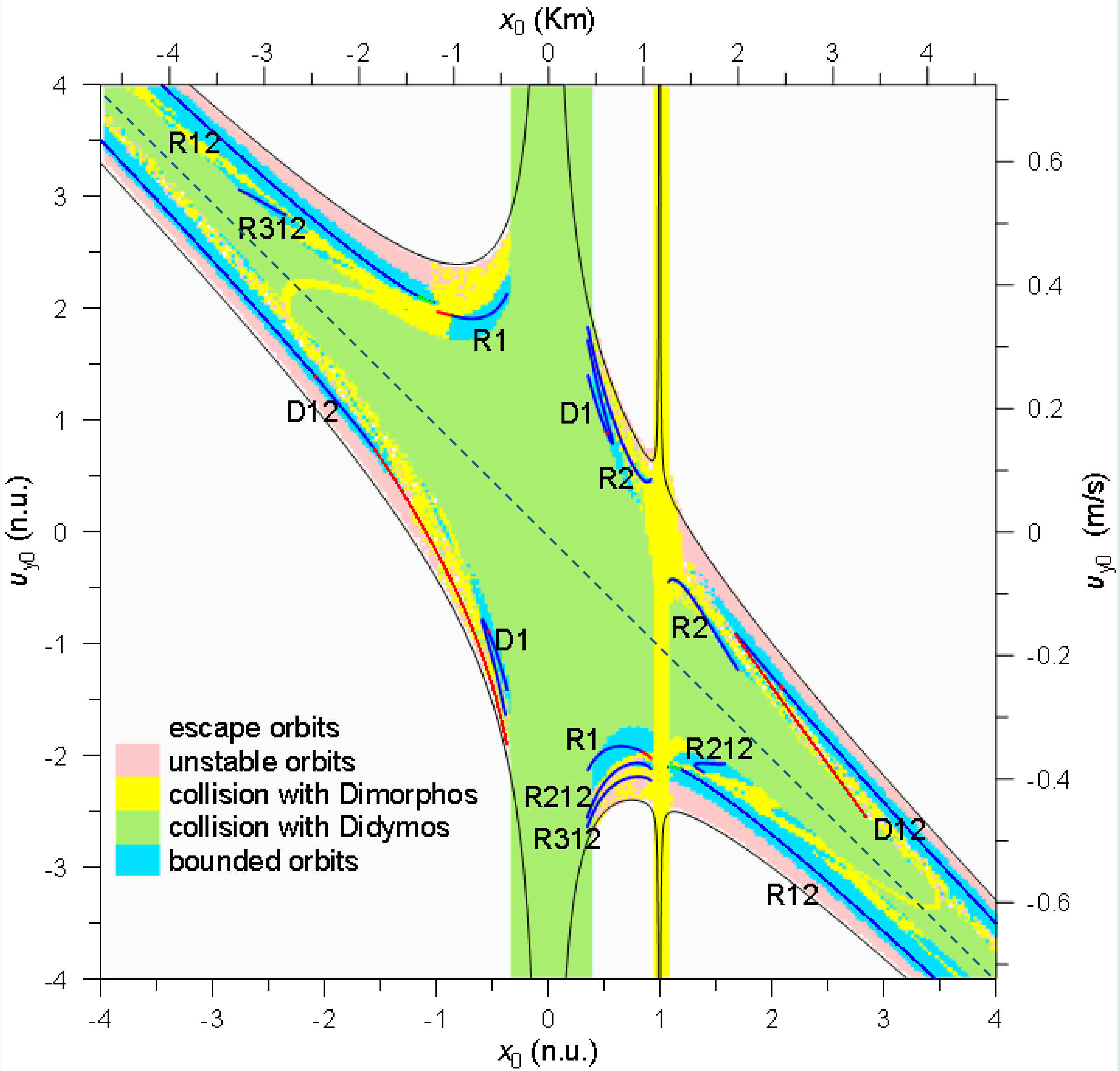


Dynamical Map for orbit classification

families of periodic orbits

- stable
- unstable





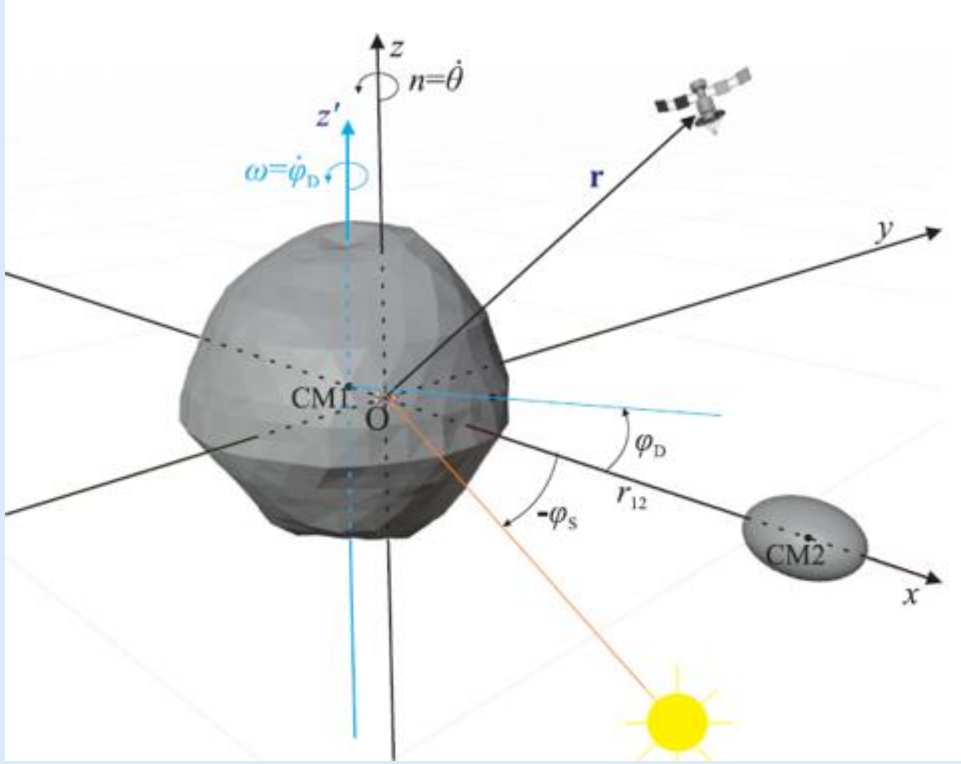
$u_{\mathrm{inertial}}=0$

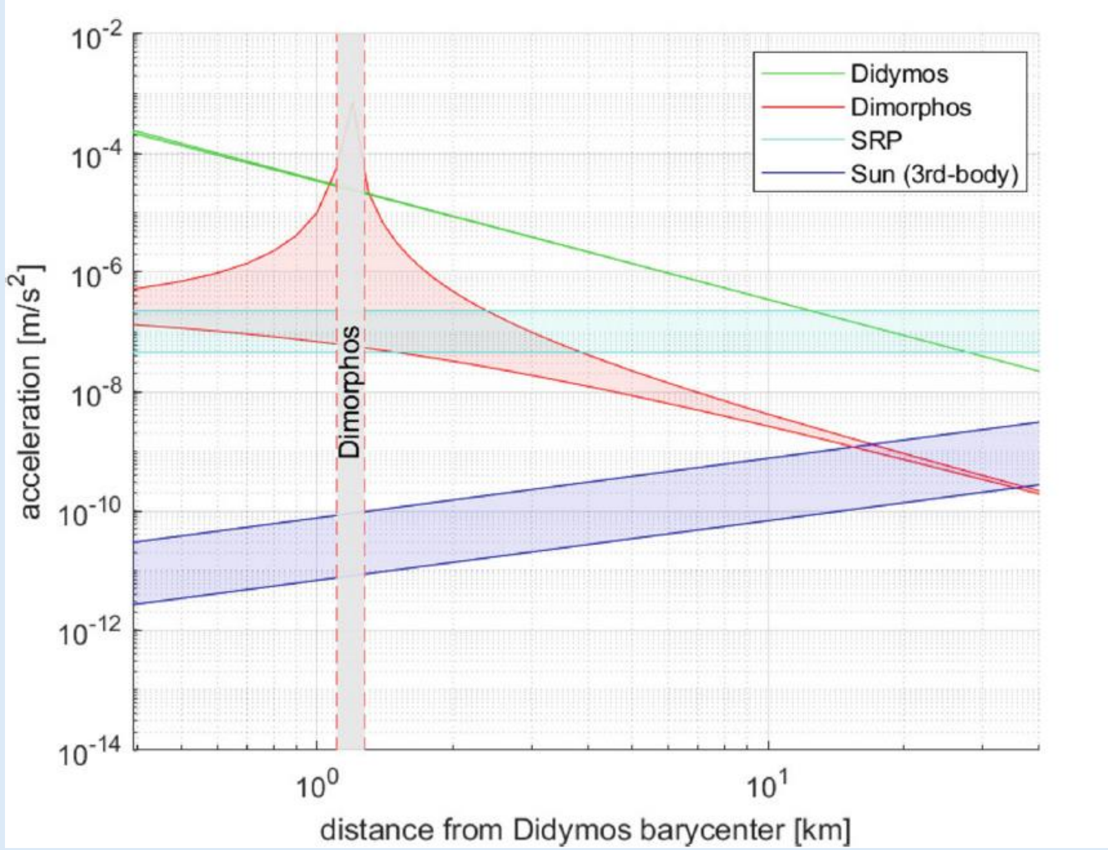
Escape estimation

$$E = \frac{1}{2}\upsilon_{inert}^2 + U_{OE} > 0.$$

A more advanced model

Ferrari et al, 2021: magnitude of accelerations

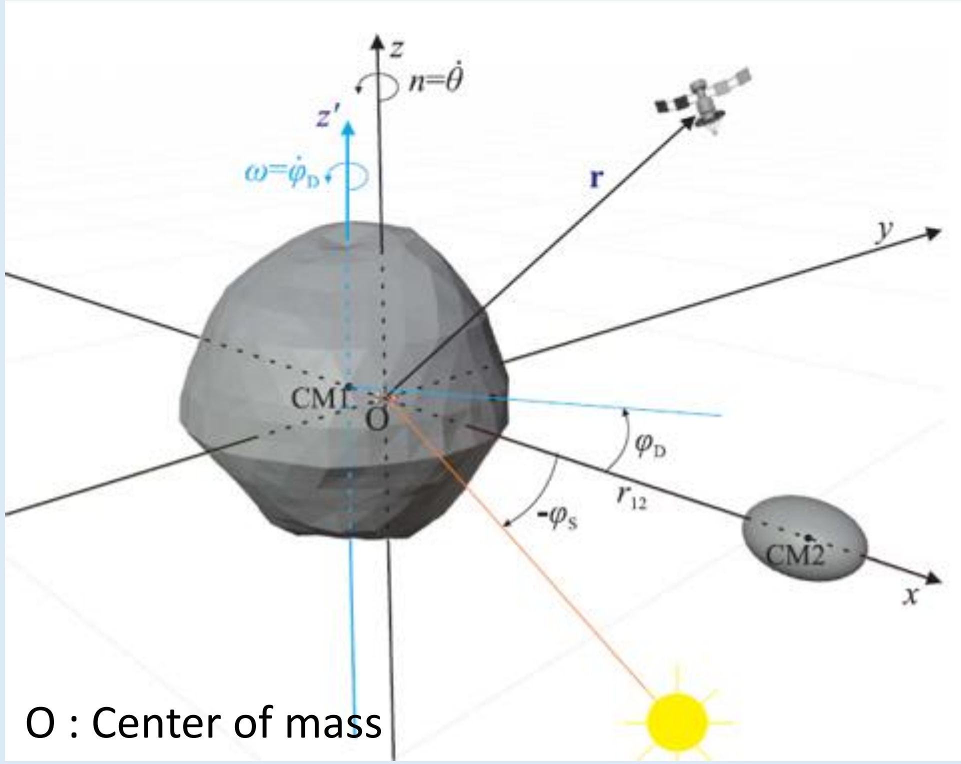




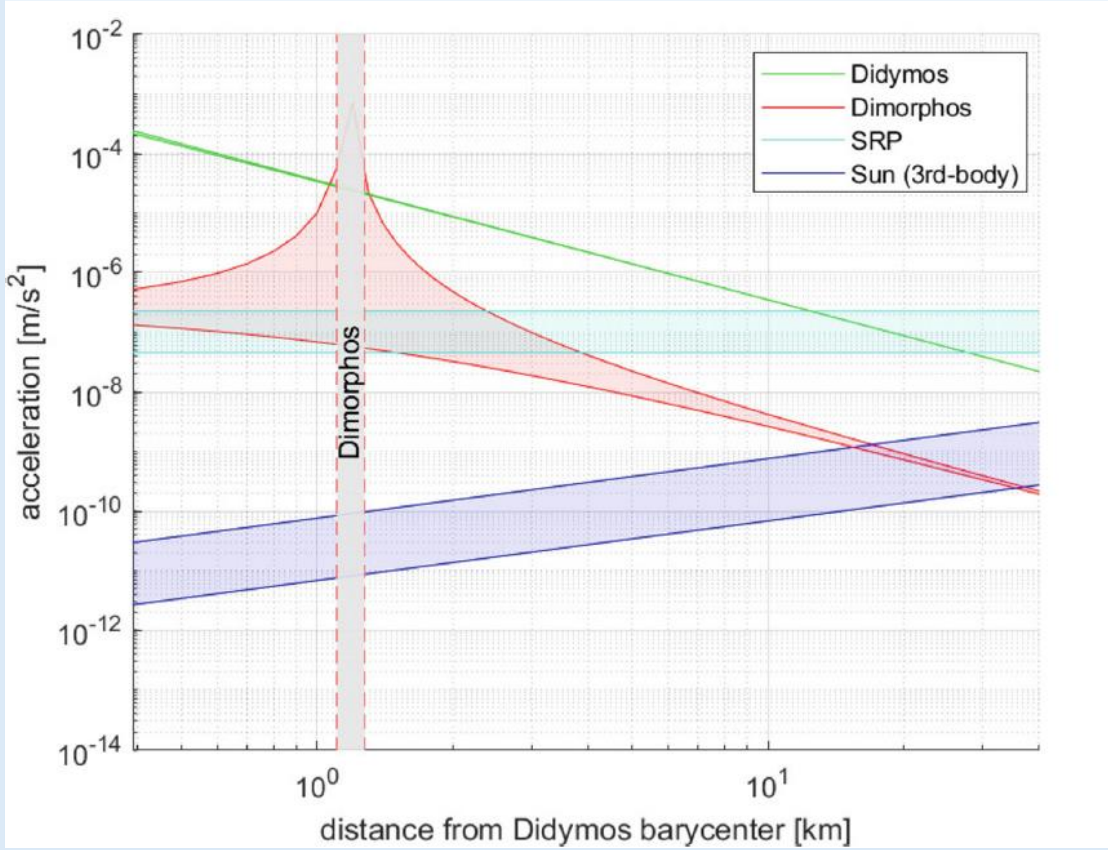
- Bodies are presented by mascons constructed by using the provided surface data (DART DRM v3)
- Fixed body frame *Ox'y'z* for the primary that rotates uniformly with angular frequency $\boldsymbol{\omega_D}$ and introduces the initial orientation angle $\boldsymbol{\varphi_D}$.
- **SRP** (cannonball model, Xin et al, 2016)

Constant magnitude F_{SRP} along the CM-Sun direction ($\boldsymbol{\varphi_S}$ angle) + Eclipse

A more advanced model

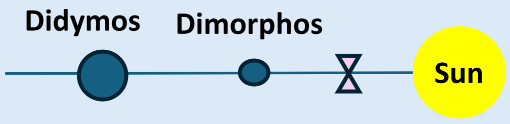


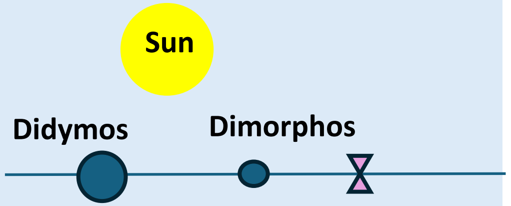
Ferrari et al, 2021: magnitude of accelerations

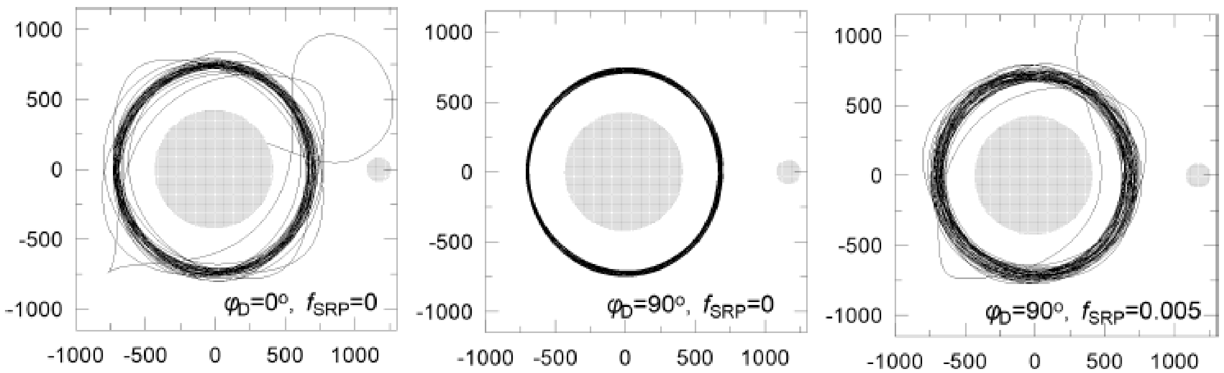


$$\ddot{\mathbf{r}} = -(1-\mu)\nabla U_1(\mathbf{r},t) - \mu\nabla U_2(\mathbf{r},t) + \left(s a_{SRP} + a_{SGR}(r)\right)\hat{\mathbf{X}}_{inert} - \dot{\boldsymbol{\omega}}_H \times \mathbf{r} - \boldsymbol{\omega}_H \times \boldsymbol{\omega}_H \times \mathbf{r} - 2\boldsymbol{\omega}_H \times \dot{\mathbf{r}}$$

$$U_i(\mathbf{r},t) = U_i(\mathbf{r},t) = -\sum_{k=1}^{N_i} \frac{1}{|\mathbf{r} - \mathbf{A}(\varphi)\mathbf{r}_k|}, \quad \varphi = \omega_D t + \varphi_D, \ \mathbf{A} = \mathbf{T}\mathbf{R}\mathbf{T}^{-1}$$





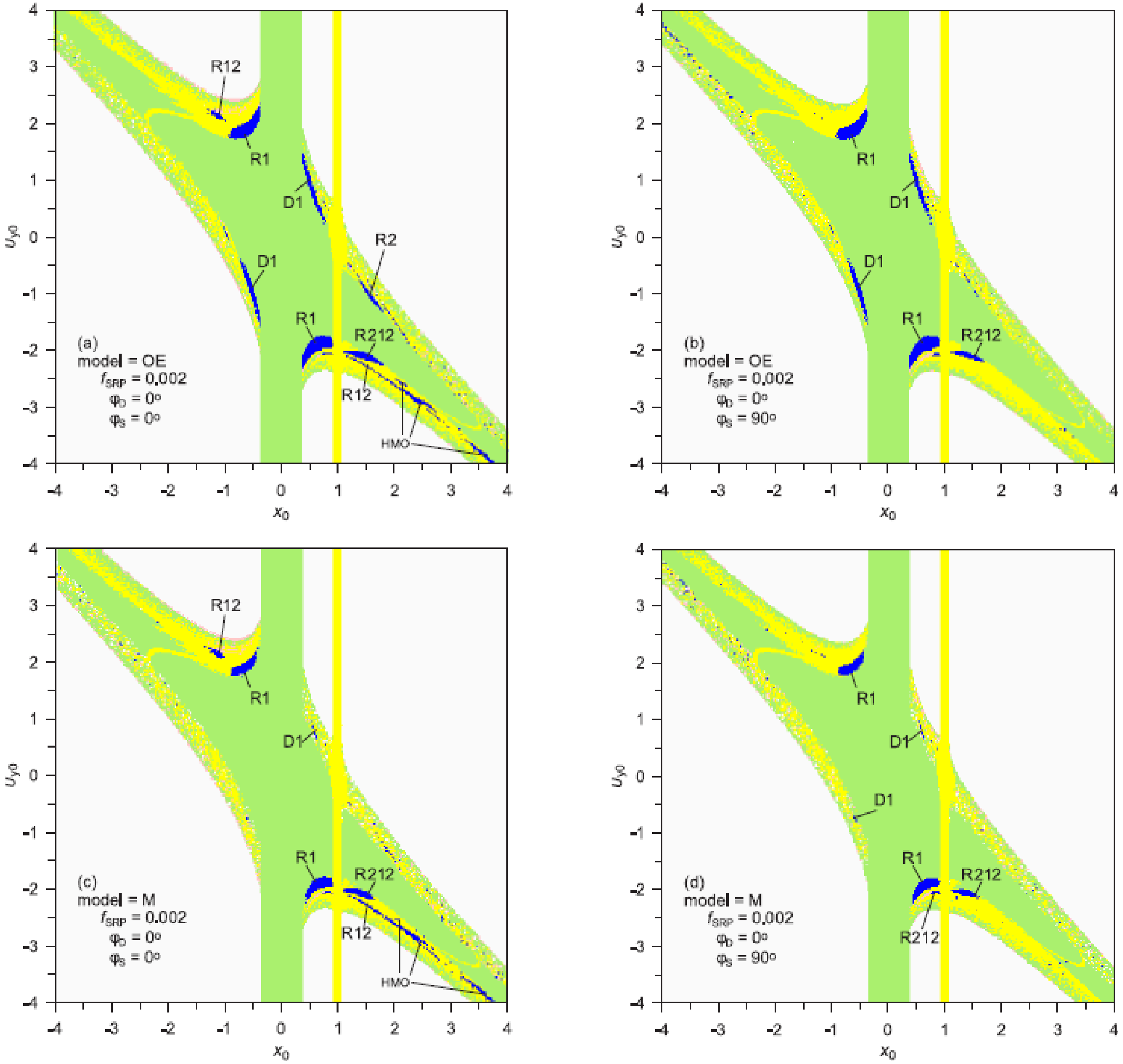


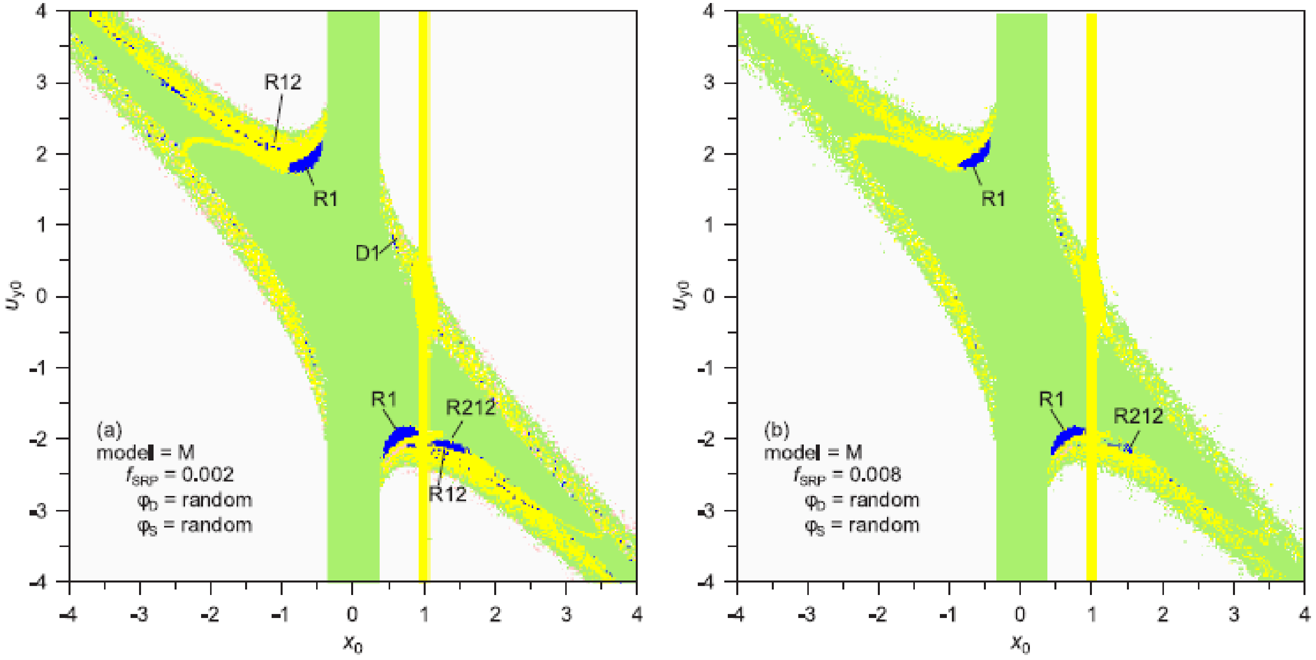
- Dependence of dynamics on initial phases φ_s and φ_D

$\varphi_S(0) \in [0, 2\pi)$

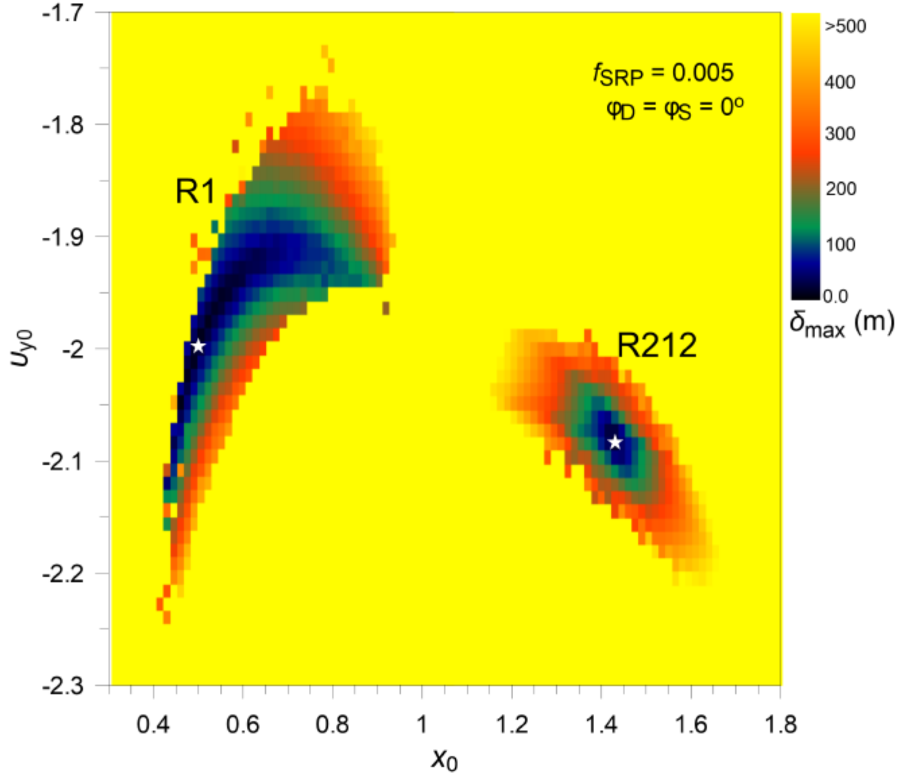
$\varphi_D(0) \in [0, 2\pi)$

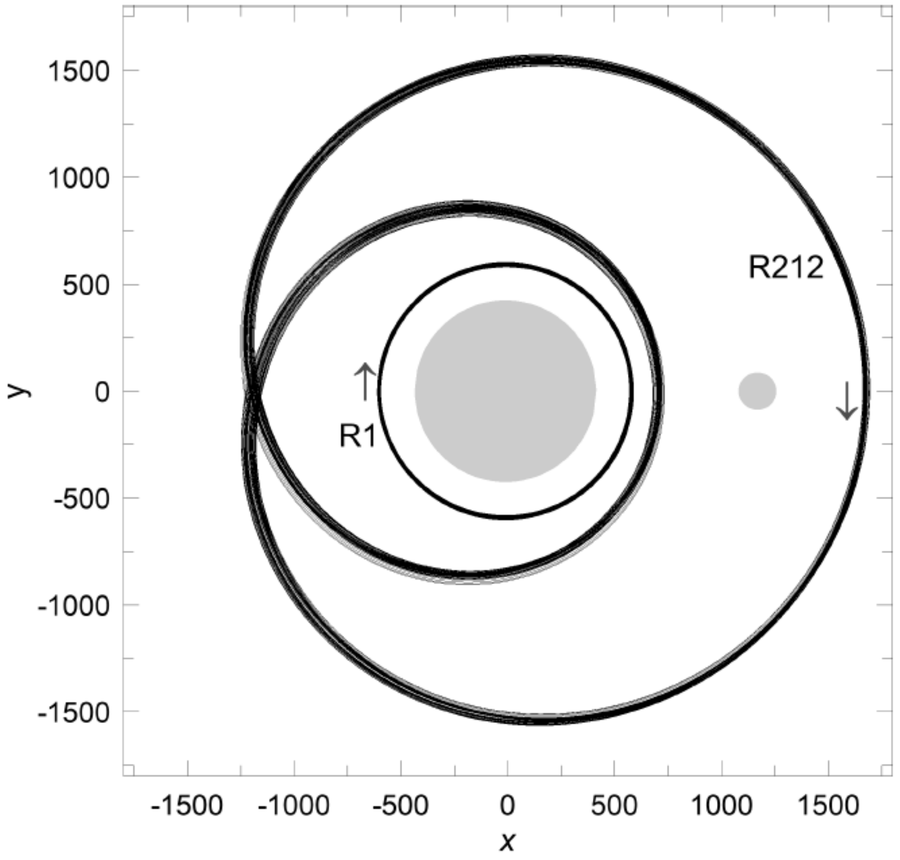
Random selection for each I.C. in the map





Almost periodic stable orbits





R1 stability : **120<*d*<350m** above the Didymos surface
Best solution ***d**=165m**

R212 stability exists for $-90^o < \varphi_S < 90^o$

Bibliography

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